Comparison of Hartmann analysis methods

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Analysis of Hartmann–Shack wavefront sensors for the eye is traditionally performed by locating and centroiding the sensor spots. These centroids provide the gradient, which is integrated to yield the ocular aberration. Fourier methods can replace the centroid stage, and Fourier integration can replace the direct integration. The two—demodulation and integration—can be combined to directly retrieve the wavefront, all in the Fourier domain. Now we applied this full Fourier analysis to circular apertures and real images. We performed a comparison between it and previous methods of convolution, interpolation, and Fourier demodulation. We also compared it with a centroid method, which yields the Zernike coefficients of the wavefront. The best performance was achieved for ocular pupils with a small boundary slope or far from the boundary and acceptable results for images missing part of the pupil. The other Fourier analysis methods had much higher tolerance to noncentrosymmetric apertures. ©2007 Optical Society of America

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1. Background

Measurements of optical wavefronts are now becoming popular, spreading from optical manufacturing to astronomy, and more and more for ophthalmology. A common wavefront sensor is the Hartmann–Shack device, where the wavefront is sampled by a lenslet array, providing an array of focal spots on a camera. From such an image, it is possible to obtain the wavefront’s slopes, by calculating the spot movement. The next stage is integration of the final wavefront from these slopes. Both steps are calculation processes that have been widely described either in the image or in the Fourier domain. For Fourier analysis, it is better if the lenslets are along a Cartesian grid, and the number of lenslets and the number of detector pixels are both maximized. Fourier methods also allow very long focal lengths of the lenslets, improving the sensitivity, because adherence of the spots to the lenslet locations is no longer required. The only condition, similar to centroiding, is that the wavefront and its derivatives are well behaved, and beams do not cross each other before the detector (the adiabatic condition).

To recap and understand the mathematical basis of wavefront reconstruction methods, we write the intensity of the spots in the Hartmann–Shack detector, in a flat wavefront case, as a regular grid,

\[ I_0(x, y) = \sum_{m,n} a_{m,n}(\cos 2\pi mx/P + \cos 2\pi ny/P), \]  

(1)

where \( P = 2\pi/k \) is the lenslet pitch (in pixel units). To develop the retrieval methods, the spots are assumed circularly symmetric and equal, so not all the harmonics are necessary. We also borrow the name sidelobes for these harmonics, as in the similar superheterodyne demodulation method. The first harmonics in \( x \) and \( y \) provide us with enough information to retrieve the wavefront, because they represent the spots layout but not their internal structure. Hence, in Eq. (1), it is possible to work with \( m = n = 1 \):

\[ I_0(x, y) \approx \cos kx + \cos ky. \]  

(2)

If the wavefront contains some aberrations, the spots are shifted by a quantity directly related to the aberrations. These aberrations modify the periodicity of the array:

\[ I(x, y) = \cos[kx + FW_x(x, y)] + \cos[ky + FW_y(x, y)], \]  

(3)

where \( W_x \) and \( W_y \) are the phase \( x \) and \( y \) derivatives that we want to determine in order to retrieve the final wavefront \( W(x, y) \) after the integration. \( F \) is the
focal length of the lenslets. In centroiding, Eq. (3) is sampled at the lenslet locations only, whereas here we assume that the slopes are continuous even between the measured spots.

It is rather easy to isolate these terms of Eq. (3) in the Fourier transform of the spot pattern, as they are contained in only four symmetric sidelobes [Fig. 1(b)]. The centers of the sidelobes are at distance \( k \) from the Fourier origin (in the discrete representation), as there are \( k \) lenslets across the pupil:

\[
\mathcal{T}\{I(x, y)\} = \mathcal{T}\{\exp(-iFw_x)\}\delta(u - k, v) + \mathcal{T}\{\exp(iFw_x)\} \times \delta(u + k, v) + \mathcal{T}\{\exp(-iFw_y)\}\delta(u, v - k) + \mathcal{T}\{\exp(iFw_y)\}\delta(u, v + k),
\]

where \( u, v \) signifies the coordinates in the Fourier domain, and \( \mathcal{T}\{a\} \) is the Fourier transform of \( a \). The different harmonics \( (m, n, 0, \pm 1, \pm 2, \ldots) \) are well separated in the Fourier domain when the number of lenslets is large, and the first harmonics contain the slopes. This is exactly the property that the Fourier demodulation technique exploits in order to get the phase derivatives. In this technique, the calculations of the slope components are performed by a rigid translation of the whole Fourier transform of the image in order to place the first sidelobe, corresponding to the lenslet frequency \( k \), in the center. To remove all other sidelobes, a low-pass filter multiplies the centered sidelobe, obtaining the filtered and centered Fourier transform of the image

\[
\mathcal{T}\{I_*(x, y)\} = \mathcal{T}\{\exp(-iFw_i)\}\delta(u, v)
\]

and similarly for the \( y \) direction. After isolation, the \( x \) and \( y \) sidelobes are inverse transformed separately, and their arguments provide the horizontal and vertical slopes. As opposed to the centroiding method where the slope is given only at the sites of the lenslets, here the slope components are found (interpolated) for each pixel in the original frame.

We chose to calculate the final wavefront in the Fourier domain as well, by employing the fact that a Fourier transform of a function is known if its derivatives in \( x \) and \( y \) are known:

\[
\mathcal{T}\{W(x, y)\} = -\left[iu\mathcal{T}\{W_x(x, y)\} + iv\mathcal{T}\{W_y(x, y)\}\right]/(u^2 + v^2).
\]

Then, to get the final wavefront, an inverse Fourier transform is used. We actually employed a similar least-squares solution, where \( u \) and \( v \) in Eq. (6) are replaced everywhere by \( \sin u \) and \( \sin v \). This accurate method can be slow, because to process one image, it is necessary to perform six Fourier transforms.

In some application, such as in adaptive optics, speed is important. In an effort to provide faster solutions, it was realized that the demodulation (but not the integration) can be performed directly on the Hartmann–Shack pattern in a manner similar to Fourier analysis [Eqs. (1)–(5): The Fourier shift of the \( x \) lobe to the center can be obtained by multiplying the original pattern by an exponential phase \( F^x(x, y) = I(x, y)\exp(-2\pi mx/P) \), and similarly for \( y \). The low-pass filtering can be performed next by a convolution of \( F^x \) with a kernel of the size of pitch \( P \). Alternatively, a low-pass filter can be performed by smoothing the same array to interpolate the (complex) values between the spots. In both methods, convolution and smoothing (or interpolation below), the slope components are the arguments of the resulting arrays. To retrieve the final wavefront, some type of integration is needed, so Eq. (6) is also applicable, requiring three Fourier transforms. Another type of integration could be useful, such as modal approximation, say by Zernike polynomials. However, Eq. (6) ensures Laplace’s property, whereas in modal calculation, only low-order terms are extracted from the wavefront slopes, and from them the wavefront is retrieved.

Another method was developed in order to minimize the calculation time, the fast Fourier demodu-
from them, the Zernike description of the wavefront. 

Comparison with the calculation of the Zernike integrates these Fourier modes to yield the wavefront.

directly from the Hartmann–Shack pattern), then in-
calculates the Fourier modes of the slope (albeit di-
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calculation is achieved without leaving the Fourier
integration of the slopes [Eq. (6)], so the whole cal-

To obtain the slopes in the Fourier domain is by ex-

ting the program was to see how this method (and

A MATLAB program was written with the theoret-
ical basis of all methods. Each image was processed
by the four methods to yield four final wavefronts as
well as the centroid result. This program was run for
images with circular apertures (ocular pupils), which

There are some important issues to stress that are
related to the processing. Theoretically, the method
to obtain the slopes in the Fourier domain is by ex-
traction of the anti-Hermitian part. To put this into
practice, it is important to take into account the fact
that an anti-Hermitian matrix has an antisymmetri-
cal real part and a symmetrical imaginary part. Then
it is possible to obtain a matrix with these properties
from the matrix, which holds the Fourier transform
of the image. Toward this end, a center of symmetry
needs to be well defined after shifting the sidelobe to
the Fourier origin. Using that center of symmetry,

Two conditions have to be fulfilled in order to get
the anti-Hermitian part, both related to the require-
ment for a center of symmetry. The first one is that
the aperture is symmetrical within the frame to be
processed. Otherwise, it is not possible to define a
center of symmetry, and the slopes cannot be re-
trieved. The second condition is that the ocular pupil
must not be cut by the optics or by low eyelids, again
losing symmetry.

If these conditions are not fulfilled, the results from
the fast Fourier demodulation will only be qualita-
tively acceptable. Thus we had to find the pupil inside
the image, defining the diameter and location of a box
holding the centered image. First, the image is low-
pass filtered in the Fourier domain, leaving only the
central lobe but excluding the spot-related lobes (a
smoothing method in the image plane is just as good,
provided it obliterates the spot pattern). After that,
the edge of the resulting image is calculated. Finally,
a circle is fitted to this edge. This circle is defined as
the pupil, and from it, the containing box is readily
calculated (Fig. 2). With this subroutine, two objec-
tives are achieved: the unimportant information out-
side the pupil is discarded, and the remaining image
is centroisymmetrical. Of course, all this is based on the
assumption that the pupil is indeed round.

A second issue is the subtraction of the reference.
As every system has some aberrations and system-
atic errors in it, a reference image is first taken
through the system from a model of a perfect eye,
processed and stored. To get the real ocular wave-
front, it is necessary to subtract from the processed
wavefront this reference wavefront. For fast Fourier
demodulation, the direct subtraction of the reference
wavefront can be performed in the Fourier domain
because of the linearity of Fourier transforms. How-
ever, it is also important to take into account the
average derivative phase generated by the shift of the
real from the reference image. This shift can arise

This information is exactly what is needed in the
integration of the slopes [Eq. (6)], so the whole calc-
ulation is achieved without leaving the Fourier domain. Thus, for every image, only two Fourier

This allows extraction of the anti-Hermitian part of
the filtered and centered Fourier sidelobes, where all
the information about the transform of the slope com-
ponents is contained:

\[
\mathcal{F}\{F(x, y)\} = \mathcal{F}\{\exp(iFW)\},
\]

so the Fourier transform of the wavefront after the
demodulation described before could be written as

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\[
\mathcal{F}\{iFW(x, y)\} = [\mathcal{F}\{\exp(iFW)\} - \mathcal{F}^*\{\exp(iFW)\}] / 2.
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from the fact that the two patterns, the reference and the final one, had different centers: the grid of spots is the same, but the pupil can appear in various locations as the eye moves, and cut out different spots. In such a case, the spot pattern can look as if shifted by the difference in the centers. If this is not removed, then an average derivative is added, and after integration, the final result for fast Fourier demodulation is totally different from the other methods.

Another point is related to the edges. It has been shown that extending the edges, by copying the peripheral spots outside the pupil, is a technique that can lower the boundary-related errors. This has been used for all the methods that we tested except for centroiding. As we show in the results, this technique is less beneficial for fast Fourier demodulation, which seems to be more sensitive to small extension errors. We believe that extrapolating the edge by a true lenslet pitch and not by integer pixels, as we do now, could remove this problem. Furthermore, the true lenslet pitch is calculated from the perfect reference image. For eyes with large edge slopes, the common result of defocus, astigmatism or spherical aberration, the actual local pitch is different from that of reference pitch, and spot extrapolation can harm the results.

Finally, aliasing effects must be considered. Except for centroiding, there is at least one Fourier stage in the calculation, where aliasing can reduce the quality of the final results. Zero padding is used to remove these effects: a matrix of zeros of twice the pupil size is defined. The image, resulting from the centering subroutine, is planted in the center of the matrix of zeros. The size of the matrix, even if necessary, slows all Fourier calculations.

3. Comparison

We now present results for two types of image, all processed with circular apertures: computer-created images and real images. For this last case, we processed both round and cut pupils and checked if those theoretical assumptions, described before, were correct and sufficient. Finally, we expressed the final wavefronts retrieved in each method in Zernike terms and calculated their first 30 coefficients to perform a quantitative comparison of all the methods and to study the reliability of fast Fourier demodulation.

Hartmann–Shack images were simulated by creating wavefronts with Gaussian distribution and a power spectrum of $-\frac{5}{2}$ to $-\frac{5}{3}$. These wavefronts were then properly propagated through a lenslet array, and their power spectrum was taken with Poisson noise added to it (no speckle was assumed). In other simulations, the Airy patterns at the foci of the lenslets were replaced by equal-magnitude delta functions [Fig. 1(a)]. Processing results are shown in Figs. 1(c)–1(g) for one of the ten realization runs. Also, the
wavefronts were retrieved with the centroiding program, and the results that we show were obtained after the subtraction of a reference image, generated by the same simulation program with much weaker aberrations. The results are similar in all the methods. There is a very small tilt difference in the full Fourier demodulation and centroiding, and the two Fourier demodulations have a slightly narrower low-pass filter. We calculated the rms differences between the original wavefront and the results, compared with the rms of the wavefront for the low noise case. The ratio was less than 7% for the convolution and smoothing methods, 9% for the Fourier method, and 10% for the centroid and full Fourier method. These numbers are influenced by the slope and bandwidth differences, and the slightly smaller centroid diameter and are accurate to $\sim 3\%$. Overall, we can conclude that, for the first time to the best of our knowledge, fast Fourier demodulation was applied successfully for circular apertures.

We now turn to real images. We show that fast Fourier demodulation works properly for whole, uncut pupils. Figure 3(a) displays the reference and the experimental data. The result of the centering subroutine and the final image to be processed (with later zero padding) are also given in Fig. 3(b). The final panels compare the final wavefronts for all the methods. The images have been processed for a 3.5 mm pupil. The results are similar qualitatively, despite edge errors that have to do with the extension of the edges and with pupil centering.

We also processed images where the 4.2 mm pupil was cut by the camera (Fig. 4). In this case, as expected from the asymmetric amplitude, the results of the full Fourier demodulation are worse. On the other hand, the shape of the wavefront is similar for all the methods, if the sign is not taken into account. We show this as an example: it happens because the program locates automatically the brightest sidelobes, and sometimes it finds an equal sidelobe in the opposite quadrant. This can easily be remedied by forcing it always to choose the sidelobes in the same quadrants.

To perform a quantitative comparison, a set of experimental images was processed in all the methods, and the final wavefronts were calculated as coeffi-
cient for 30 Zernike polynomials. The main coefficients were plotted together. The results for defocus, spherical, vertical astigmatism, and vertical coma for different sizes of the pupil are shown in Fig. 5 for pupils that vary from 2.7 to 3.5 mm, and the results are similar for all the methods. For bigger pupils, the results deviated, as is visible for the spherical aberrations in the 3.5 mm pupil. In the wavefronts shown in Fig. 3, some edge effects were also present. We suspect that they arise from the fact that for large pupils fast Fourier demodulation becomes even more sensitive to precise extension of the boundary and to accurate centroiding. The bigger the pupil, the more lenslets are visible, and the higher the frequency \( k \) of the sidelobes. The same errors in the spot locations lead to larger relative errors for denser spots. In addition, aperture centroiding becomes more sensitive to pixelation errors, and comparison with the reference wavefront becomes less accurate. We plan to probe into these error sources and remove them, for example, by shifting the images by noninteger pixel steps when extending their edges.

We also ran comparisons on the calculation time for all the methods. For Hartmann–Shack experimental images, where the image size is 768 \( \times \) 576 pixels, the processing time of fast Fourier demodulation, similar to an interpolation method and slightly lower than a convolution method, is half of the Fourier demodulation. This method employs two Fourier transforms, as compared with the six by a simple demodulation method. The balance is due to the fact that in using MATLAB, the fast Fourier transform routines are written efficiently in C as compared with the rest of the processing.\(^5\)

As a conclusion, the fast Fourier demodulation performs well with actual ocular circular pupils. Its results are currently accurate for round pupils having well-behaved boundary slopes. Otherwise, all the methods yield similar results, with differences of less than 10% among them. These differences occur mostly at the boundary, where the extrapolation of spots by the reference pitch, rounded to integer pixels, might not match the actual ocular pitch at the edges. We now develop the next analysis version to deal with this problem as well as the occasional shifting of the pupil across the reference image.

The final choice of algorithm depends now on the application: for adaptive optics, either the smoothing or the fast Fourier method will be acceptable; for accurate wavefront sensing, the slower full Fourier method or the convolution with a large kernel.

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References