Hartmann Fourier analysis for sensing and correction

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Abstract
The Hartmann pattern lends itself naturally to Fourier analysis, providing directly mirror commands. Slopes are integrated without returning to the image domain. We modeled, simulated and tested these algorithms on two separate adaptive optics systems.

Introduction. Hartmann-Shack (HS) wave front sensors are very common in the fields of wave front sensing and for controlling adaptive optics systems. For simple adaptive optics systems, where the number of lenslets is low, it is possible to deduce directly from the movements of the HS centroids what are the commands for the deformable mirrors. However, in cases where the number of lenslets is very large, or for high precision measurements, it seems that centroids calculation is not the most efficient method. We developed a number of methods to (a) calculate the wave front slopes from the HS pattern, (b) calculate the wave front itself from its slopes, (c) calculate the control commands directly in the Fourier domain, and (d) calculate the wave front itself from the HS in the Fourier domain.

Fourier analysis. We consider the HS pattern as a grid of $\delta$ functions whose transform is

$$ I_0(x, y) = \sum_{m,n=0} a_{m,n} \left[ \cos(kx + FW_x) + \cos(ky + FW_y) \right]. \quad (1) $$

The grid period, or the lenslet pitch, is $P = \frac{2\pi}{k}$ and the spots are moved according to the gradient components $W_x$ and $W_y$ of the wave front $W(x, y)$. We neglect all high harmonics which contain information about the shape of the spots. Writing the cosines as sums of exponentials and transforming, one gets for the first harmonics only

$$ \tilde{I}(u,v) \approx \delta(u-k) + \delta(u+k) \quad (2) $$

where $\tilde{I}$ or signifies a Fourier transform of $I$, etc. Thus we see that each side lobe, at distance $k$ from the Fourier origin, carries the Fourier transform of the wave front slope.

Simple demodulation. From Eq. 2 it is clear that to get the gradient, all one has to do is take the vicinity of each side lobe, down-shift it to the origin, and inverse-transform it to the image domain, to get $\exp \pm kW_x$ and $\exp \pm kW_y$ (Fig. 1). The arguments of these two quantities are the sought slopes $FW_x$, $FW_y$. Care should be taken to avoid aliasing in the first Fourier transform, e.g. by employing an empty array of double the size, into which the HS pattern is copied before the transform (zero padding).

Convolution. The process of Fourier demodulation can also be performed without reverting to Fourier transforms: shifting in frequency, say by $k$ in the $x$ direction, is equivalent to multiplying the HS pattern by $\exp -ikx$. Then choosing the neighbourhood of the side lobe near the Fourier origin is equivalent to smoothing or low-pass filtering of the image. The filter kernel is of the size of the lenslet pitch $P$.

Smoothing. More saving in time is possible if one employs another smoothing scheme. The method we have used was a simple shift-and-add scheme: if the smoothing kernel is, say, 1, 2, 1, then by shifting the (now complex) array by half the pitch and adding double that array to the original array would achieve this fit. This can be repeated at one-quarter the pitch, and so on.
**Borderline cases.** The round edge of the HS pattern means a jump in the processed slopes. A solution to this problem was found by assuming that the slope is constant at the edges, or that a closed integral which includes the boundary is zero. We were able to achieve this by duplicating the last HS spots exactly one period $P$ outside the aperture.

**Fourier control.** It is customary to use the $x$ and $y$ slopes of the wave front as an input to the reconstructor matrix. In the Fourier scheme this is performed in either one of two ways, corresponding to the chosen process:

**Side lobs.** As the complex values around the lowest side lobses describe the components of the transform in full, they are equivalent to the values of the slopes as described in Section 2 (Fig. 2). Only one Fourier transform is necessary for adaptive optics, without any centroiding.

**Sampled wave front.** The result of convolution or smoothing is two frames holding the two components of the wave front slope, at the detector resolution (Fig. 1), which can now be sampled at the actuator locations. We have indeed constructed such a control system, where the control commands are taken from few pixels inside the wave front slopes, and were able to correct variable wave front aberrations in the lab.

**Wave front reconstruction.** The reconstruction is performed in the image domain, based on least-squares fitting of the wave front or modes thereof to the given slopes. We employed direct or least squares fitting of the data in the Fourier domain. Given the slopes $W_x$ and $W_y$, it is possible to show that the wave front is given by

$$W(u,v) = \frac{uW_x(u,v) + vW_y(u,v)}{u^2 + v^2}, \quad (3)$$

and the boundary conditions, on a round aperture, are imposed iteratively. Based on a least squares solution, one switches $u$ and $v$ to $\sin u$ and $\sin v$, or $2\sin u/2$ and $\sin v/2$ using sine and cosine transforms.

**Fourier full reconstruction.** When the wave front is ergodic, namely slopes are well-behaved and not too high, it is possible to perform both demodulation and integration in the Fourier domain. The number of transforms reduces from six to two, and phase dislocations are rare. The results are comparable to the other methods (Fig. 3). The noise content of the different methods was also compared to the centroid method and shown to be superior.
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References