Enhancement of superconductivity by local inhomogeneities

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We study the effect of inhomogeneity of the pairing interaction or the background potential on the superconducting transition temperature, \(T_c\). In the weak coupling BCS regime, we find that inhomogeneity, which is incommensurate with the Fermi surface nesting vectors, enhances \(T_c\) relative to its value for the uniform system. For a fixed modulation strength we find that the highest \(T_c\) is reached when the characteristic modulation length scale is of the order of the superconducting coherence length.

\[ \mathcal{H} = \mathcal{H}_0 + \mathcal{H}_U \]

\[ \mathcal{H}_0 = \sum_{k\sigma} \xi_k c_{k\sigma}^\dagger c_{k\sigma} \]

\[ \mathcal{H}_U = - \sum_r U(r) n_{\uparrow}(r) n_{\downarrow}(r), \]

where \(\xi_k = E_k - \mu\) and \(n_{\sigma}(r) = c_{\sigma}^\dagger(r) c_{\sigma}(r)\) is the occupation number of electrons of spin \(\sigma\) at position \(r\). Within this model, our goal is to understand whether for a fixed average pairing strength \(U(r)\), a uniform or nonuniform \(U(r)\) yields a higher transition temperature, \(T_c\). In the weak-coupling limit, we can derive the BCS condition for the onset of superconductivity from the Hamiltonian (1),

\[ \Delta_q = \int \frac{d^p}{(2\pi)^p} U(q-p) K(p) \Delta_p, \]

where \(\Delta_q = \sum_k U(k) \langle c_{q-k/2+p} c_{q-k/2-p} \rangle\), \(U(k)\) is the Fourier transform of the pairing interaction, and \(K(p)\) is the pairing kernel. The pairing kernel depends on temperature \(T\) and the mean-field (MF) superconducting transition is defined by the temperature at which the integral equation has a nontrivial solution. The kernel can be calculated from the normal state electron Green functions\(^{11}\)

\[ K(p) = N_f \ln \left[ \frac{2\gamma \omega_D}{\pi \sqrt{T^2 + (v_f p)^2}} \right] \Theta(\omega_D - |v_f p|), \]

where \(N_f\) is the density of states at the Fermi surface, \(v_f\) is the Fermi velocity, \(T\) is temperature, and \(\ln \gamma = 0.577\) is Euler’s constant. Here we also introduced an explicit high-energy cutoff for the attraction, \(\omega_D\). For \(T \approx v_f p\) this expres-
The modulation of the pairing interaction leads to the mixing between Cooper pairs with different center-of-mass momenta. For simplicity we first assume a harmonic modulation \( Q = 2\pi/L \) of the pairing, \( U(r) = \bar{U} + U_0 \cos(Q \cdot r) \). In this case the integral equation (2) reduces to a system of linear equations \( \Delta_n = \bar{U} \Delta_{n+1} = \Delta_{n-1} + (U_0/2)[K_{n-1} \Delta_{n-1} + K_{n+1} \Delta_{n+1}] \), where \( \Delta_n = \Delta(nQ + q_0) \) and \( K_n = K(nQ + q_0) \). The “parent” momentum \( q_0 \) defines the minimal momentum of a Cooper pair in the connected family \( \Delta(nQ + q_0) \).

The paring instability occurs at the temperature \( T_c \), such that the largest eigenvalue of matrix \( \mathcal{M} \) is equal to 1. In the uniform case, \( U_q = 0 \). Consider the \( q_0 = 0 \) family and without loss of generality, take \( U_q > 0 \). Since all of the matrix elements of \( \mathcal{M} \) are non-negative, by Perron’s theorem, the maximal eigenvalue is a positive number that is larger than any diagonal matrix element, including \( \bar{U} \).

Thus, generically, the superconducting onset temperature \( T_c \) is increased whenever \( U_q \neq 0 \).

This result can be understood from an analogy with a quantum mechanical particle in a tight-binding chain. Defining a variable \( \Delta_n = \bar{U} \Delta_{n+1} \), the BCS condition takes a simple symmetric form, \( \Delta_n = (1/\bar{U}) \Delta_{n-1} - (U_0/2\bar{U})(\Delta_{n+1} + \Delta_{n-1}) \). The “hopping” term delocalizes the particle and thus reduces the “energy” below its minimal on-site value \( 1/\bar{U} \).

Clearly, this leads to a relative increase of \( T_c \), even in the case of multi-\( Q \) modulation.

**Large \( Q \) limit.** In this limit, the quickly oscillating coupling is ineffective at mixing different modes, so that the off-diagonal terms in \( \mathcal{M} \) are rapidly decaying with \( n \). We are then justified in keeping only a small portion of the matrix surrounding the \( n = 0 \) term. The lowest order correction to the homogeneous result is obtained by considering couplings between \( \Delta_0 \) and \( \Delta_1 \). The largest eigenvalue in this case is

\[
\lambda_{\text{max}} = \frac{U}{2} \left[ K_0 + K_1 + \sqrt{(K_0 - K_1)^2 + \frac{2K_0 K_1 U_0^2}{U^2}} \right].
\]

Given the separation of energy scales, \( T_c \ll v_f Q \ll \omega_D \), we obtain \( T_c \) by solving \( \lambda_{\text{max}} = 1 \).

**Electron density modulation.** Before going further, let us in parallel consider the case of homogeneous coupling \( U(r) = \bar{U} \), with inhomogeneity caused by a background potential variation. In the simplest case of the harmonic modulation, the additional contribution to the Hamiltonian (1) is

\[
\mathcal{H}_p = \rho \sum_{i\sigma} c_i^\dagger c_{i\sigma} \cos Q \cdot r_i.
\]

It is easy to see that for particle-hole symmetric density of states (DOS), the linear in \( \rho \) contributions to the BCS instability condition equations vanish identically. For small values of \( Q \), the modulation acts as a slowly varying shift in the local chemical potential with the amplitude proportional to \( |\rho| \). Thus, we only get a linear in \( \rho \) contribution for asymmetric DOS, \( N(\epsilon) = N_f + N'(-\epsilon - \epsilon_F) \). We then find that the BCS equations are identical to the case of inhomogeneous pairing interaction with the modulation strength

\[
U_0^{\text{eff}} = - \frac{\bar{U} N_f \rho}{N_f}.
\]

**Ginzburg-Landau analysis \( Q \xi \ll 1 \).** We now consider the general case of slow variation of the pairing strength and/or background potential. The Ginzburg-Landau free energy functional in the presence of inhomogeneity is

\[
F = - \int \rho(r) \Delta(r)^2 + \frac{\rho_0}{2} \int \Delta(r)^2 + \frac{\beta}{2} \int \Delta(r)^4.
\]

Here we assumed that the order parameter remains real even in the presence of inhomogeneity. We include both the coupling of the superconducting order parameter to a density wave, as well as the inhomogeneity of the pairing interaction. For small amplitude modulation of the pairing interaction, \( U(r) = \bar{U} + U_0 \cos(Q \cdot r) \) with \( |\bar{U}/U(r)| < 1 \), the two mechanisms are formally equivalent. For particle-hole asymmetric DOS, from the above considerations, \( \alpha = -\bar{U} \rho N_f \). For simplicity, we only consider the inhomogeneous \( U(r) \) case here. The pair susceptibility kernel is given by Eq. (3). In the long wave length limit,

\[
K(r - r') = \delta(r - r') N_f \left[ \ln \frac{2\gamma \omega_D}{\pi T} + \xi^2 \nabla^2 \right],
\]
sition, the cubic terms in Eq. (9) can be neglected. Expanding in $U_0/\bar{U}$, and transforming Eq. (9) to Fourier space, we obtain a system of equations connecting $\Delta(k)$ and $\Delta(k\pm Q)$. For small $Q$, $\Delta(k\pm Q)\approx \Delta(k)$, and after expanding up to the second order in $Q$, we obtain

$$-g_{\text{max}}\Delta(k) = (\xi k)^2 \Delta(k) - \frac{1}{2} k^2 Q^2 \partial_k^2 \Delta(k).$$

(10)

Here $g_{\text{max}}$ denotes $g(r)$ evaluated at $U(r)=\bar{U}+|U_0|$ and $A=\sqrt{U_0/(N_f\bar{U})}$ (note that there is no explicit constraint on the value of parameter $A$ since it is a ratio of two small numbers). The MF transition temperature is determined by the smallest eigenvalue of the differential operator on the right-hand side. This eigenvalue corresponds to the “ground-state energy” of a harmonic oscillator, $\xi Q A/\sqrt{2}$. The corresponding transition temperature $T_c=\gamma_{\text{max}}^{-1} \exp(-\xi Q A/\sqrt{2})$, is only slightly less than the transition temperature $T_{c,\text{MF}}$ for a system with a homogeneous pairing interaction $U_{\text{max}}=\bar{U}+|U_0|$. More importantly, it is easy to see that in the limit of small $Q\xi$, the order parameter is exponentially suppressed in the region of smaller pairing interaction relative to its value at the peak,

$$\Delta_{\text{min}} \sim \Delta_{\text{max}} \exp(-A(\xi Q)^{-1}).$$

(11)

This, in turn, implies that phase fluctuations, which we discuss below, can reduce the global phase coherence temperature significantly below $T_{c,\text{MF}}$.

The expression of Eq. (11) is only valid at the MF transition temperature, where $\Delta$ is infinitesimal. To determine $\Delta$ below $T_{c,\text{MF}}$ we need to solve the nonlinear Eq. (9). In a $d$-dimensional superconductor, with an arbitrary smooth variation of $U(r)$, the boundary of the “classically forbidden” region, $g(r)>0$, is a $(d-1)$-dimensional surface. Hence, near the boundary the problem is essentially one-dimensional, and in the $g(r)>0$ region we can apply the standard WKB approximation to solve the linearized Eq. (9). The prefactor is fixed by matching the WKB solution to the intermediate asymptotic at the boundary $x_p$, which can be obtained by solving the full Eq. (9) in a linear potential $g(x)=g'(x_0)(x-x_0)$. We then find that in the particular case of harmonic modulation discussed above, the order parameter distance $d$ away from the boundary is approximately

$$\Delta(d) \sim T\sqrt{A Q \xi} \exp(-A \sqrt{Q\xi}(d/\xi)^{3/2}).$$

(12)

This expression is obtained assuming $g'(x_0)=A^2 Q$, and therefore valid only for the temperatures sufficiently below $T_c \sim \gamma_{\text{max}}^{-1}$. So long as $T \gg \gamma_{\text{min}} N_f U_0$ (where $\gamma_{\text{min}}$ is the uniform $T_c$ of a system with pairing strength $\bar{U}-|U_0|$), the distance from the turning point to the minimum point of $\Delta$ is $d \sim L$. Notice that this expression depends on temperature not only explicitly, but also through $\xi=v_f/T$.

Phase fluctuation effects (still with $Q\xi \ll 1$). A consequence of the large spatial variations in the mean-field $\Delta(r)$ is that fluctuation effects are severe where $\Delta(r)$ is small. Of these, the most important are the thermal fluctuations in the phase of the order parameter, i.e., such that

$$\Delta(r) = |\Delta_{\text{MF}}(r)| e^{i\theta(r)}$$

where $\Delta_{\text{MF}}(r)$ is the solution of Eq. (9), and $\theta(r)$ is a slowly varying function of $r$. The free energy cost of such phase fluctuations is $F_\theta=\int dr \langle\nabla \theta\rangle^2$, with the local superfluid stiffness $\langle\nabla \theta\rangle=\xi^2 \partial_n \Delta_{\text{MF}}/r^2$. In general, the phase-ordering temperature estimated using this as the effective Hamiltonian is reduced from the mean-field transition temperature [i.e., the temperature at which $\langle\nabla \theta\rangle$ vanishes], but by an amount that depends on dimensionality, and on the spatial arrangement of the regions of suppressed stiffness.

For concreteness, we consider the case of a two-dimensional (2D) superconductor. At finite temperature, no true long-range order is possible. However, at $T < T_{KT}$, binding of topological excitations into vortex-antivortex pairs leads to a state with quasi-long-range order, which has a nonzero superfluid stiffness. While for homogeneous BCS superconductors in 2D, the difference between MF and the Kosterlitz-Thouless (KT) transition temperatures is tiny, $(T_c^{\text{MF}}-T_{KT})/T_c^{\text{MF}} \sim T_c^{\text{MF}}/T_F$ (where $T_F$ is the Fermi temperature), for inhomogeneous superconductors, the suppression of $T_{KT}$ is generally much larger. For a smooth random distribution of $J(r)$, an estimate of $T_{KT}$ can be made based on the effective superfluid density,

$$T_{KT} \sim \left[\langle J(r) \rangle [1/\langle J(r) \rangle]^{-1}\right]^{1/2} \sim \left[J_{\text{min}} J_{\text{max}}^{1/2}\right].$$

(13)

This expression has a particularly transparent meaning for the unidirectional “stripedlike” variation of $U(r)$ that we treated explicitly when solving the mean-field equations above. There, $J_{\text{max}}$ corresponds to the stiffness along the stripes and $J_{\text{min}}$ perpendicular to the stripes. The corresponding anisotropic XY model directly leads to the result Eq. (13). In this case, we find...
Together with Eq. (12) for $\Delta_{\text{min}}(T_{KT}) \sim \Delta(L,T_{KT})$, this equation implicitly defines $T_{KT}$. With logarithmic accuracy we find that $T_{KT} \sim \min(T_{c}^{MF}, v_{F}Q/l_{A})$.

In any case, barring certain artificial geometries, it is clear that for a long wave length modulation, $\xi \sim 1$, the Kosterlitz-Thouless temperature $T_{KT}$ is exponentially lower than the MF transition temperature; on the other hand, for modulations with $\xi \gg 1$, the phase fluctuation region is very narrow and $T_{KT} \approx T_{c}$. In this regime, the mean-field superconducting temperature is still exponentially enhanced relative to its value in the uniform state with the same average paring interaction strength, $\bar{U}$. For even faster modulation, $\xi \gg 1$, the MF transition temperature drops since the pair-breaking interaction strength, $U_{\bar{v}}$, also increases with $\xi$. This trend is presented qualitatively in Fig. 1.

For a “dirty” superconductor with a mean free path shorter than the clean coherence length, $\ell = v_{F}\tau < \xi$, the effect of phase fluctuations can be estimated in the same way as in the clean limit, with minor changes in prefactors but with the coherence length redefined as $\xi_{c} = \sqrt{\ell}$. 

**Summary.** We studied the effect of nanoscale inhomogeneity on the superconducting transition temperature, $T_{c}$. We considered two possible kinds of inhomogeneity: the modulations of the paring strength and of the background potential. In the weak coupling BCS regime, we find that inhomogeneity, which is incommensurate with the Fermi surface nesting vectors, enhances $T_{c}$ relative to its value for the uniform zero center-of-mass momentum pairing. For a fixed modulation depth we find that the highest $T_{c}$ is reached when the modulation wavelength is of the order of the superconducting coherence length. For shorter wavelengths, the superconductor cannot take advantage of the locally favorable conditions, while for the longer wavelengths, the global superconductivity is suppressed due to the phase fluctuations on the weak links. Similar results also apply to unconventional superconductors in the presence of smooth (on the $1/k_{F}$ length scale) inhomogeneities. Clearly oversimplified, the presented picture bears resemblance to the high-temperature superconducting cuprates, where considerable experimental evidence indicates that the maximum $T_{c}$ occurs at a crossover between a regime where $T_{c}$ is controlled by the pairing scale and where it is a phase-ordering transition. Evidence for inhomogeneous superconductivity has also been found in several other materials, including $\text{Ba}_{1-x}\text{K},\text{BiO}_{3}$, $\text{WO}_{3}$, and Pt.

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