Onset of Dissipation in Zener Dynamics: Relaxation versus Dephasing

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Received September 13, 1990

We study the onset of dissipation in an externally driven small system, coupled to an environment. More specifically, we focus on the interplay between Zener dynamics (induced by the driving source), and the interaction with the environment, which gives rise to both dephasing and relaxation. We first consider a toy model, consisting of an externally driven two-level system coupled to various types of environments. We demonstrate the onset of dephasing, manifested as a decay of interference terms. In addition, we derive an effective equation of motion for the density matrix of the system, in the presence of a dissipative coupling to a thermal bath characterized by a broad spectrum. We extend our analysis of dephasing and relaxation to deal with the dynamics of a multi-level system. We show that dephasing and relaxation have competing effects on the dynamics. Dephasing tends to enhance the absorption of energy into the system, due to destruction of localization in the energy space; relaxation, on the other hand, gives rise to a loss of energy, thus limiting the rate of energy absorption. We calculate the dependence of the conductance on the external bias and find that it is nonmonotonous due to the interplay between diffusion and relaxation in energy space. In the weak dephasing regime we obtain fluctuations in the conductance, due to quantum interference effects. Our results can be applied to one-dimensional conducting rings threaded by time-varying Aharonov–Bohm flux. © 1991 Academic Press, Inc.

1. INTRODUCTION

Since the early days of solid state physics, the response of conductors to external driving sources, and in particular the onset of irreversibility, have been issues at the focus of attention. Early attempts to describe resistance in metals have been made by Drude [1]. His theory is based on the assumption that electrons in metal behave like ideal classical objects. Transport properties are determined by a phenomenological parameter, which describes the mean free time between successive collisions. The Drude theory has been later improved by Sommerfeld [1] taking into account the fact that electrons in metal obey the Fermi–Dirac statistics. This correction resolved the discrepancy between the Drude model and some experimental results (in particular, low temperature behavior of the specific heat and magnetic susceptibility of metals). A more general and systematic approach to the description of transport properties, employing basic ideas of the Drude theory, is formulated in terms of the Boltzmann equation. There are two important assumptions underlying this approach: the first is the assumption of a single relaxa-
tion time. The other is that transport properties at a weak external bias are described within linear response.

A more modern approach to the description of linear response has been established by Kubo and Greenwood [2]. The Kubo–Greenwood formula relates dissipation in a system to its microscopic structure. It introduces the concept of the fluctuation–dissipation theorem, which invokes the correspondence between non-equilibrium behavior of the system (the response to an external field) and statistical fluctuations at equilibrium. Evaluation of the dc conductivity by the Kubo–Greenwood formula requires one to deal carefully with order of limits: the size of the system is taken to infinity before the frequency of the external field approaches zero [3]. In view of the microscopic picture, this procedure ensures quasicontinuity of the spectrum of the system, which, in turn, is required to give rise to resonant transitions (even in the limit of zero frequency), and hence to dissipation. This approach is therefore suitable for macroscopic systems; for small systems characterized by discrete spectra, the onset of dissipation requires the explicit coupling of the system to an external reservoir.

A different approach to this issue of resistance has been put forward by Landauer [4]. According to the Landauer formula, elastic scattering from an obstacle, connected to two electron reservoirs, gives rise to resistance. The presence of reservoirs (which provide thermalization of the injected electrons), is essential to provide irreversibility, although the value of resistance is determined by the scatterer. In view of Landauer’s approach one distinguishes the mean free times for elastic and inelastic scattering: the latter does not determine the value of the resistance, but rather the rate at which the system “dissipates” its excess energy to the reservoirs and consequently thermalizes. In other words, the inelastic time determines the range of bias for which linear response is valid. The Landauer picture has been related to the Kubo–Greenwood approach [5]. The Landauer picture may be used as a starting point to derive what is known as localization theory in one dimension (1D) [6]. Localization occurs due to subtle quantum interference effects in disordered media [7]. Notwithstanding this quantum coherent picture, in a realistic experimental setup, coupling to extra degrees of freedom at finite temperatures gives rise to destruction of phase memory. Phase breaking events are often described in the literature by a phenomenological parameter $\tau_\phi(L_\phi)$, a typical time (length) scale over which quantum coherence is maintained [8–9]. In the presence of a finite $L_\phi$, the size of the system plays an important role. One should distinguish the behavior of small mesoscopic systems, whose linear size $L$ satisfies $L \ll L_\phi$, from the behavior of larger systems. In the latter, transport in a disordered medium can be described as a diffusion process; the macroscopic diffusivity is determined by the quantum picture on a scale $L_\phi$.

The interest in mesoscopic physics have brought back to focus the issue of dissipation and irreversibility. When addressing a finite (closed) system, one should avoid the assumption of a quasi-continuous spectrum, and rather consider a discrete spectrum. An external driving source may be included as a time-dependent term in the Hamiltonian. Such a coupling induces transitions among the instan-
taneous (adiabatic) energy levels of the system (Zener transitions); this introduces a mechanism for pumping energy into the system [10–15]. It has been argued by Landauer [16], that the related dynamics is reversible, hence the energy pumped into the system through Zener transitions is, in principle, retrievable. In view of this observation, the energy pumped into a system via Zener dynamics should not be interpreted as dissipation. Landauer's observation was demonstrated for a single Zener transition. More recently it has been shown [17] that when many consecutive Zener transitions are involved, the time required to undo them (within certain accuracy) is exponentially large. Although this may suggest that, in practice, Zener dynamics alone leads to dissipation, the whole issue has not yet been resolved. Another possible approach to irreversibility in small systems, which is compatible with many realistic situations, involves the introduction of coupling to external degrees of freedom. Such an external environment, assumed to be a macroscopic system, may provide a relaxation mechanism and absorb energy out of the system at hand. The inclusion of an external environment has been implemented phenomenologically by Debye [18], and later employed by Landauer and Büttiker [19–20] to account for dissipation in small metallic loops, threaded by a magnetic flux. Within this approach one invokes a relaxation time $\tau_{eq}$, where $1/\tau_{eq}$ describes the exponential relaxation rate of the density matrix of the system towards thermal equilibrium. Landauer and Büttiker have assumed that the rate at which energy is being pumped by the source into the system (this energy is eventually being dissipated to the heat bath) is adiabatic, and neglected Zener transitions [21]. In a later work, Gefen and Thouless [14] (see also Refs. [11, 13, 15]) have accounted for Zener transitions, identifying two stages in the dissipation mechanism: in the first stage energy is being pumped into the system by the driving source, whereas in the second—this energy is being transferred to the environment through inelastic events. The former is dominated by Zener dynamics; it has been shown that as long as quantum coherence is maintained in the system—interference effects lead to localization in the energy direction [14, 15]. Hence, in the absence of inelastic events, the long time average of the energy pumped into the system reaches saturation.

The works alluded to above describe the coupling to external environment by means of a phenomenological parameter. Many recent works employ instead a microscopic approach, according to which the degrees of freedom of the environment and the coupling are all described as terms in a microscopic Hamiltonian. Coupling of a two-level system to a bath of harmonic oscillators, with the coupling term linear in the coordinates of the oscillators, has been extensively studied [22–24]. In particular, the effect of the bath on the Zener transition probability has been calculated [23, 24]. Some of these works employ ideas and techniques due to Feynman and Vernon [25], developed later by Caldeira and Leggett [26]. Feynman and Vernon have introduced the influence functional, which expresses the influence of the environment on the system in terms of the coordinates of the system only. Along with energy dissipation, the coupling of a system to external degrees of freedom gives rise to a phase breaking mechanism (dephasing) [27–29]. One may
describe dephasing employing a microscopic approach. A recent work by Stern et al. [30] demonstrates the equivalence of two physical interpretations of the influence functional—either as a trace left by the system on the environment, or as an uncertainty in the quantum mechanical phase. In view of the latter one identifies dephasing with the attenuation of quantum interference terms. This effect is not necessarily associated with energy exchange between the system and environment.

In the present work we study the onset of dissipation in a small systems, coupled to both an external driving source and an environment. The dynamics of the system is dominated by the interplay between Zener transitions, induced by the driving source, and the interaction with the environment, which gives rise to both dephasing and relaxation. Dephasing is manifested as suppression of interference terms; relaxation to thermal equilibrium with the environment is provided by exchange of energy between the system and the environment, induced by the coupling. We distinguish these two aspects of the coupling to the environment; moreover, we demonstrate that they have competing effects on the dynamics. This competition can be traced back to the fact that, on one hand, dephasing tends to destroy quantum coherence, which, in turn, is responsible to the localization in the energy direction. This destruction of localization effects leads to enhancement in the rate at which energy is being pumped into the system. On the other hand, relaxation of the system, facilitated by its coupling to the environment, gives rise to a loss of energy, thus limiting the rate of energy absorption into the system [31]. To enable a microscopic description of the various types of environments, we introduce and analyze a toy model. We derive explicit expressions for the attenuation of interference terms and study its dependence on the type of coupling to the environment. In addition, we show that coupling to a thermal bath with a broad spectrum, may lead to an additive term in the equation of motion for the density matrix of the system, of a form similar to the phenomenological relaxation employed by Landauer and Buttiker [19]. This relaxation term is obtained when the coupling to the bath induces interlevel transitions. We then incorporate both dephasing and relaxation in the dynamics of an externally driven system, whose spectrum consists of a multitude of energy levels. Dephasing is included as an exponential decay of the off-diagonal density matrix elements, while relaxation—as a decay of the entire density matrix towards thermal equilibrium with the bath. Our analysis may shed light on the dynamics of a variety of small driven systems, in particular 1D metallic rings threatened by a time-dependent magnetic flux. We find that, for any finite relaxation rate, the system evolves in time through a transient behavior, after which it eventually reaches a steady state. In this steady state, the energy pumped into the system by the driving sources is balanced by the energy dissipated to the bath. The competition between dephasing and relaxation is manifested as an enhancement of the steady state time averaged energy for stronger dephasing, as opposed to a reduction in the average energy for a shorter relaxation time. In the steady state it is possible to define the power dissipation in the system (and thus resistance), as the rate of energy transfer into the system (which is equal to the rate of energy loss to the bath). We obtain the conductance as a function of the external bias for various
parameter regimes. We find a non-monotonic behavior of the conductance, dominated by the interplay between diffusion and relaxation in energy space. We note that, in the limit of small bias, linear response does not hold. In the regime of weak dephasing, we also observe quantum fluctuations in the conductance, which modulate the smooth behavior determined by the diffusion picture.

The outline of the paper is as follows. In Section 2 we present a toy model, describing a two-level system whose dynamics involves two consecutive interlevel transitions. In this section we analyze this toy model in the absence of coupling to external degrees of freedom. We evaluate the asymptotic interlevel transition probability (denoted by $P$, cf. Eq. (2.14)). We also find conditions for which Eq. (2.14) reduces to a simple form (Eq. (2.12)). In the latter form, the transition probability $P$ is expressed as an interference pattern of two partial waves. This section is appended by Appendices A and B, which include the technical details related to the derivation of Eqs. (2.12) and (2.14). In Section 3 we discuss the onset of dephasing due to the coupling of our toy model to an environment. We distinguish between a non-dynamical and a dynamical environment. Non-dynamical environments are considered in Section 3(a), and are modeled by a bath of two-level degrees of freedom. We derive expressions for the attenuation factor ($F_d$), which multiplies the interference term, and for the dephasing time $\tau_d$. The main results are presented in Eqs. (3.9), (3.10), and (3.13), while technical details of the derivation are summarized in Appendices C and D. We also argue that a non-dynamical environment prepared in a thermal state will have the same effect on the system as when prepared in a coherent state. This is not the case for a dynamical environment. The latter is represented by a set of harmonic degrees of freedom, and is discussed in Section 3(b). We derive an expression for $F_d$ (Eq. (3.19)), which differs from the one derived in the non-dynamical case. Details of our analysis are summarized in Appendix E. In Section 4 we consider various microscopic models for dissipation (relaxation) in our toy model. In Section 4(a) we consider the effect of a coherent environment. Our results (Eqs. (4.2) and (4.3)) manifest the role played by energy exchange between the system and the environment. Details of the derivation are given in Appendix F. In Section 4(b) we discuss coupling to a thermal bath and derive an effective equation of motion for the density matrix (cf. Eq. (4.20)). We obtain explicit expressions for the relaxation rate to thermal equilibrium with a bath of either fermions (Eq. (4.11)) or bosons (Eq. (4.12)). Technical details are included in Appendix G. In Section 5 we apply our results for dephasing and relaxation to a more complicated, externally driven multi-level system, described in terms of a time-dependent Hamiltonian (cf. Eq. (5.2)). We employ a master equation description of the time evolution of the density matrix (cf. Eqs. (5.3(a)) and (5.8)). We solve these equations numerically and obtain the expectation value of the energy as function of time (Fig. 5). In addition, assuming that the model describes electrons in a metallic ring, we plot the single particle conductance (Fig. 6) and the total conductance of the ring (Fig. 7) as a function of the externally induced e.m.f. Our numerical results are supported by approximate analytical expressions for the conductance, based on a diffusion picture in energy level space (cf. Eq. (5.14(a)))
We distinguish between the behavior in the strong dephasing regime (where the diffusion coefficient is given by Eq. (5.14(b))), and the weak dephasing regime (with the diffusion coefficient given by Eq. (5.19)). Analytical estimates for the conductance in both regimes are given in Eqs. (5.24), (5.25), and (5.26); details of the derivation are summarized in Appendix H. In Section 6 we present a brief summary of the results of Section 5, discuss the relevance to experiments, and mention few open questions left for future study.

2. Two Consecutive Zener Transitions

As a first step of our analysis we focus on the dynamics of an externally driven system, dominated by Landau–Zener transitions. It is important to characterize effects that arise due to coherence in such a system, e.g., interference among consecutive transitions. To this end we analyze a simple toy model, containing some of the essential features of realistic, more complex systems. The model was previously introduced and discussed in the context of atomic collisions [32]; in the present work, we concentrate mainly on the manifestation of coherence effects in this model.

We consider the time-dependent Hamiltonian

\[ H_0(t) = \left( \frac{\alpha t^2 - \varepsilon}{2} \right) \sigma_z + \left( \frac{\Delta}{2} \right) \sigma_x. \]  

(2.1)

Here

\[ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

are the spin-$\frac{1}{2}$ Pauli matrices, and $\varepsilon > 0$. The corresponding instantaneous (adiabatic) energy levels are

\[ \varepsilon_{u(d)} = \pm \frac{1}{2} \sqrt{\Delta^2 + (\alpha t^2 - \varepsilon)^2}; \]

(2.2)

in Fig. 1, $\varepsilon_{u(d)}$ are plotted as functions of the parameter $t$. The energy gap of the adiabatic spectrum (i.e., $\varepsilon_u - \varepsilon_d$) is minimal at $t = \pm \sqrt{\varepsilon/\alpha}$ (see Fig. 1), which satisfy $(\alpha t^2 - \varepsilon) = 0$; we refer to these points as "narrow gaps." We expect that the main contribution to the interlevel transitions will come from the neighborhood of these narrow gaps. The wave function of the system can be written in the form

\[ \psi(t) = C_-(t) \exp\{i\phi_-(t)\} \mid - \rangle + C_+(t) \exp\{i\phi_+(t)\} \mid + \rangle, \]

(2.3)

where the diabatic states $\mid - \rangle$ and $\mid + \rangle$ are defined by $\sigma_z \mid \pm \rangle = \pm \mid \pm \rangle$, and $\phi_{\pm}(t) = \pm \int^t (\alpha u^2 - \varepsilon) \, du/2\hbar$ is the dynamical phase accumulated by the states $\mid \pm \rangle$ in the absence of the coupling term $(\Delta/2)\sigma_x$. Imposing the initial conditions $C_-(\infty) = 1$, $C_+(\infty) = 0$, the asymptotic transition probability, $P$, is defined by

\[ P = |C_+(\infty)|^2. \]

(2.4)
Below we calculate $P$ and show that the resulting expression is dominated by an interference effect. This interference emerges from a splitting of $\psi(t)$ into two partial waves: one which undergoes a transition (Zener tunneling) from $| - \rangle$ to $| + \rangle$ in the vicinity of $t = -\sqrt{\varepsilon/\alpha}$, and the other, in the vicinity of $t = \sqrt{\varepsilon/\alpha}$. These correspond to the two trajectories, $a$ and $b$, in the energy-time scheme, depicted in Fig. 1. In Section 3 we demonstrate the destruction of this interference due to interaction with external degrees of freedom.

Below we derive an explicit form for $P$. We obtain a rather cumbersome expression. This reflects the fact that the two trajectories alluded to above (which correspond to the transitions at $t = -\sqrt{\varepsilon/\alpha}$ and $t = \sqrt{\varepsilon/\alpha}$, respectively), may actually be nonseparable, and the two transition events may overlap in time. We note that there are two important time scales in the problem. One ($\tau_z$) characterizes the span in time of a single Landau–Zener transition [33]. The other refers to the separation in time of two consecutive Zener attempts; in our case this separation time, $t_s$, corresponds to the distance between the two narrow gaps:

$$t_s = 2\sqrt{\varepsilon/\alpha} \quad (2.5)$$

(cf. Fig. 1). Overlap between two consecutive transition events becomes important for $t_s \lesssim \tau_z$.

Before we outline the derivation of $P$, it is instructive to present an approximate evaluation of this quantity, expected to be valid in the limit $\tau_z \ll t_s$. In this limit we assume that the description of the problem in terms of two well-separated
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trajectories is meaningful. This picture is indeed borne out by an explicit calculation of $P$. We consider the dynamics in the neighborhood of each narrow gap independently. In the vicinity of the points $t_{\pm} = \pm \sqrt{\varepsilon/\alpha}$, we linearize the time-dependent term of $H_0(t)$. The asymptotic [34] transition probability from one level to another is then given by the Landau–Zener expression [35]

$$P_{LZ} = \exp \left( -\frac{\pi A^2}{2\hbar \alpha_{\text{eff}}} \right), \quad \alpha_{\text{eff}} \equiv \left( \frac{d(\alpha t^2 - \varepsilon)}{dt} \right)_{t = \pm \sqrt{\varepsilon/\alpha}} = 2 \sqrt{\varepsilon/\alpha}. \quad (2.6)$$

In order to keep track of the phase of the wavefunction it is useful to calculate the transition amplitude. A single Landau–Zener transition may be described by means of a scattering matrix $S$,

$$\begin{pmatrix} C_-(t_f) \\ C_+(t_f) \end{pmatrix} = S \begin{pmatrix} C_+(t_i) \\ C_-(t_i) \end{pmatrix}, \quad S = \begin{pmatrix} \tilde{\tau} & \tilde{\tau}^* \\ \tilde{\tau}^* & -\tilde{\tau} \end{pmatrix}. \quad (2.7)$$

Here $t_i \ll t_\pm - \tau/2$ and $t_f \gg t_\pm + \tau/2$, where $t_\pm = \pm \sqrt{\varepsilon/\alpha}$ (in practice, $S$ is calculated setting $t_i = -\infty$ and $t_f = \infty$); $C_\pm(t)$ and $\phi(t)$ are defined following Eq. (2.3). We find

$$\tilde{\tau} = \sqrt{P_{LZ}} \exp\{i\theta_i\}, \quad \hat{\tau} = \sqrt{1 - P_{LZ}} \exp\{i\theta_r\}. \quad (2.8)$$

The phases $\theta_i$ and $\theta_r$ can be easily calculated in two extreme limits (see Appendix A): in the adiabatic limit ($A^3 \gg \hbar \alpha_{\text{eff}}$) we find

$$\theta_i = \theta_r = 0; \quad (2.9a)$$

in the sudden limit ($A^3 \ll \hbar \alpha_{\text{eff}}$) we find

$$\theta_i = 0 \quad \text{and} \quad \theta_r = -\pi/4. \quad (2.9b)$$

The asymptotic components of the wavefunction, as it evolves through the two narrow gap spectrum, are given by

$$\begin{pmatrix} C_+(\infty) \\ C_-(\infty) \end{pmatrix} = S \Delta S \begin{pmatrix} C_+(-\infty) \\ C_-(-\infty) \end{pmatrix}, \quad \Delta = \begin{pmatrix} \exp\{i\phi\} & 0 \\ 0 & \exp\{-i\phi\} \end{pmatrix}. \quad (2.10)$$

where $S$ is given by Eqs. (2.6) through (2.9). Here $\phi$ is the dynamical phase, accumulated by the components $|+\rangle$ and $|-\rangle$ of $\psi(t)$ over the time interval $t_s$ between the two transitions:

$$\phi = \frac{1}{2\hbar} \int_{\sqrt{\varepsilon/\alpha}}^{\sqrt{\varepsilon/\alpha}} (\alpha t^2 - \varepsilon) \, dt = \frac{2\varepsilon^{3/2}}{3\hbar \alpha^{1/2}}. \quad (2.11)$$

The asymptotic transition probability $P$ (Eq. (2.4)) is given then approximately by [36]

$$P \approx 2P_{LZ}(1 - P_{LZ})[1 - \cos(2\phi - 2\theta_r)]. \quad (2.12)$$
We next present a detailed, more general solution for $P$. Although this quantity has been evaluated in various limits in Ref. [32], we sketch in Appendix B the calculation of $P$ for the sake of completeness. The calculation is performed in both the adiabatic and the sudden limits. These extreme limits, defined now for the problem at hand (rather than for a single Landau–Zener transition), are determined by the parameter

$$\delta \equiv \frac{\Delta}{(\hbar^2 \alpha)^{1/3}};$$

(2.13)

$\delta \gg 1$ ($\delta \ll 1$) corresponds to the adiabatic (sudden) limit. The final result is

for $\delta \gg 1$, \quad $P \approx \left(\frac{\pi}{3}\right)^2 \exp \{2R(\delta, s)\} \sin^2\{I(\delta, s)\},$ \quad (2.14a)

where $s \equiv \epsilon/\Delta,$

$$R(\delta, s) \equiv \frac{(\pi \delta)^{3/2}(1 + s^2)}{8} [f(s) - g(s)], \quad I(\delta, s) \equiv \frac{(\pi \delta)^{3/2}(1 + s^2)}{8} [f(s) + g(s)],$$

$$f(s) \equiv \frac{(2s)_{2F_1}(3/4, 7/4; 3/2; -s^2)}{I(3/4) I(7/4)}, \quad \text{and} \quad g(s) \equiv \frac{(2s)_{2F_1}(1/4, 5/4; 1/2; -s^2)}{I(3/4) I(7/4)};$$

for $\delta \ll 1$, \quad $P \approx (\pi \delta)^2 Ai^2\left[-\epsilon/(\hbar^2 \alpha)^{1/3}\right],$ \quad (2.14b)

where $Ai(x)$ denotes the Airy function. Note that in both limits $P$ is an oscillating function of the parameter $\epsilon$, which provides a parameterization of the separation

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**Fig. 2.** The asymptotic transition probability $P$ (cf. Eq. (2.4)) as function of the separation parameter $\epsilon/(\hbar^2 \alpha)^{1/3}$ (cf. Eq. (2.1) and Fig. 1): (a) $\delta = 4.5$, which corresponds to the adiabatic limit (Eq. (2.14a)); (b) $\delta = 0.1$, the sudden limit (Eq. (2.14b)).
between the narrow gaps (see Fig. 2). The expressions for $P$ in Eq. (2.14) are considerably simplified for large arguments of the special functions involved (see Appendix B). Asymptotic expressions for these special functions require

$$\varepsilon \gg \Delta \quad \text{for} \quad \delta \gg 1,$$  \hspace{1cm} (2.15a)

$$\varepsilon \gg (h^2 \alpha)^{1/3} \quad \text{for} \quad \delta \ll 1.$$  \hspace{1cm} (2.15b)

The conditions (2.15) are equivalent to

$$\tau_z \ll t_s,$$  \hspace{1cm} (2.16)

where $\tau_z$ is the Zener time at each narrow gap. To see this, we employ the results obtained by Mullen et al. [33]. They have estimated $\tau_z$ in both the adiabatic and the sudden limits, and found

$$\tau_z \sim \Delta/\alpha_{\text{eff}} \quad \text{for} \quad \Delta^2 \gg h\alpha_{\text{eff}},$$  \hspace{1cm} (2.17a)

$$\tau_z \sim \sqrt{h/\alpha_{\text{eff}}} \quad \text{for} \quad \Delta^2 \ll h\alpha_{\text{eff}}.$$  \hspace{1cm} (2.17b)

Equations (2.5) and (2.17), with $\alpha_{\text{eff}}$ defined in Eq. (2.6), imply the equivalence of Eqs. (2.15) and (2.16). From Eqs. (2.14) we find that the asymptotic expressions for $P$, in both the adiabatic and the sudden limits, coincide with Eq. (2.12). We conclude that if Eq. (2.16) is satisfied, the dynamics is indeed separable into two consecutive Landau–Zener transitions and a dynamical phase accumulated between them. The cosine term in the square brackets of Eq. (2.12) represents interference between consecutive transitions; hence it is expected to be suppressed in the presence of a dephasing mechanism (see Section 3).

To complete our picture, we note that the model discussed above can be mapped (in the same way as any two-level model) into a purely classical problem of a precessing polarization vector in an external field [37]. For example, a magnetic moment in a magnetic field, which in our case is time-dependent:

$$B(t) = \frac{1}{2}(\Delta, 0, (\alpha t^2 - \varepsilon)).$$  \hspace{1cm} (2.18)

The polarization of this magnetic moment is $P = \langle \sigma \rangle$, i.e., the expectation value of a vector whose components are Pauli matrices. Assuming that initially (at $t \to -\infty$) the magnetic moment was polarized along the $z$-axis (with $P_z = 1, P_x = P_y = 0$), then $(1 - P_z)/2$ at $t \to \infty$ is analogous to the transition probability $P$ in the two-level model. As long as $|\alpha t^2 - \varepsilon| \gg \Delta$, $P_z$ is nearly unchanged, and the magnetic moment performs precession about the $z$-direction with a time-dependent frequency. During intermediate times (near $t = +\sqrt{\varepsilon/\alpha}$), $P$ acquires a finite component in the $x-y$ plane. The dynamical phase $\phi$ in Eq. (2.11) is analogous to the angle traversed by $P$ in the $x-y$ plane within the time interval $-\sqrt{\varepsilon/\alpha} < t < \sqrt{\varepsilon/\alpha}$. The solution for $P$ (Eq. (2.14), or (2.12) in the separable case) implies that $P_z$ at $t \to \infty$ is an oscillating function of the $z$-component of the field (which determines the angle in the $x-y$ plane).
3. Onset of Dephasing

In this section we introduce a coupling of our toy model, described in Section 2, to external degrees of freedom. We examine a few simple microscopic models for such environments and demonstrate the destruction of phase memory (dephasing) due to coupling to an environment. For our particular system, dephasing is manifested as suppression of the interference term (e.g., Eq. (2.12)).

In general, we assume that the system and the environment are described by a Hamiltonian of the form

$$H(t) = H_0(t) + \sigma_z \hat{O}_\text{en} + H_\text{en},$$

where $H_0(t)$ is given by Eq. (2.1), $H_\text{en}$ is the Hamiltonian of the uncoupled environment, and $\hat{O}_\text{en}$ is an operator representing environment degrees of freedom. A coupling of the environment to $\sigma_z$ is expected to cause statistical fluctuations in $\epsilon$, and thus in the phase accumulated between the transitions (Eq. (2.12)). In what follows we derive a microscopic picture for this dephasing and discuss its dependence on various characteristics of the environment and coupling. We refer the reader to a detailed discussion of dephasing in systems described by time-independent Hamiltonians [30].

Following Stern et al. [30], we distinguish between a dynamical and a non-dynamical environment.

(a) A Non-dynamical Environment

A non-dynamical environment is defined by

$$[H_\text{en}, \hat{O}_\text{en}] = 0$$

(cf. Eq. (3.1)). This is the simplest scenario for the onset of dephasing. When Eq. (3.2) is satisfied, the coupling term does not induce transitions among different energy states of the environment, and therefore there is no dissipation of energy. Below we demonstrate that a non-dynamical environment can still suppress interference quite efficiently, provided that it is prepared in a state that is not an eigenstate of $\hat{O}_\text{en}$.

As an example we consider a bath of $N$ two-level degrees of freedom, with

$$H_\text{en} = -\frac{1}{2} \sum_{k=1}^{N} \zeta_k \sigma_z^{(k)} \quad \text{and} \quad \hat{O}_\text{en} = -\frac{1}{2} \sum_{k=1}^{N} V_k \sigma_z^{(k)}.$$

Here $\sigma_z^{(k)}$ denotes the Pauli matrix associated with the $k$th degree of freedom in the bath. The wave function of the system + environment is assumed to have the form

$$\Psi(t) = (C_-(t) e^{i\phi_-(t)} | - \rangle + C_+(t) e^{i\phi_+(t)} | + \rangle) \otimes \prod_{k=1}^{N} (a_k | - \rangle^{(k)} + b_k | + \rangle^{(k)}),$$

(3.4)
where $\sigma^{(k)}_{\pm} | \pm \rangle^{(k)} = | \mp \rangle^{(k)}$; the phases $\phi_{\pm}(t)$ are defined following Eq. (2.3). Note that the bath here is assumed to consists of pure quantum mechanical states (as opposed to a thermal bath). We solve the time-dependent Schrödinger equation for $\Psi(t)$, with $H(t)$ given by Eqs. (2.1), (3.1), and (3.3). After tracing over the environment degrees of freedom (see Appendix C), we obtain the asymptotic transition probability:

$$P_{\text{eff}} = |C_+ (\infty)|^2 = \sum_K \prod_{k \in K} |a_k (-\infty)|^2 \prod_{k \notin K} |b_k (-\infty)|^2 P \left\{ \epsilon - \sum_{k \in K} V_k + \sum_{k \notin K} V_k \right\}. \quad (3.5)$$

Here $P\{\epsilon - \sum_{k \in K} V_k + \sum_{k \notin K} V_k\}$ is the transition probability for the uncoupled system (given by, e.g., Eq. (2.14)), with $\epsilon$ being shifted ($\epsilon \rightarrow \epsilon - \sum_{k \in K} V_k + \sum_{k \notin K} V_k$); $K$ is a subset of $\{1, ..., N\}$ ($\sum_K$ denotes the sum over all possible choices of a subset); $a_k (-\infty)$ and $b_k (-\infty)$ determine the initial conditions for the environment; $C_-(\infty) = 1$ and $C_+(\infty) = 0$. Equation (3.5) has the general form

$$P_{\text{eff}} = \sum_v D(v) P\{\epsilon + v\}, \quad (3.6)$$

where the weight function $D(v)$ depends on the distribution of coupling coefficients $V_k$ and the initial conditions of the environment. This form reflects the fact that the environment induces uncertainty in the value of $\epsilon$ (i.e., in the time-independent part of the separation between diabatic levels; cf. Fig. 1). Had the environment been prepared in an eigenstate of $H_{\text{en}}$, one would rather observe a well-defined shift in the effective $\epsilon$. The fact that the quantum state of the environment corresponds to a superposition of eigenstates is what gives rise to this uncertainty. Before discussing this effect quantitatively, we note that the type of environment introduced here affects the transition probability $P_{\text{eff}}$ because $P(\epsilon + v)$ depends nontrivially [38] on $v$.

We now consider Eq. (3.6) first in the separable limit $\tau_z \ll \tau_s$ (see previous section). In this limit we assume that the argument of $P\{\epsilon + v\}$ is large enough for the asymptotic approximation (Eq. (2.12)) to hold, for any $v$ in the sum. We also assume weak coupling, i.e., $v \ll \epsilon$ for all $v$. Note that both of the above assumptions can be justified if $D(v)$ is vanishingly small outside a certain finite range of $v$'s around $v = 0$. To lowest order in $v/\epsilon$, the $v$-dependence in Eq. (2.12) can be neglected in $P_{\text{LZ}}$, but not in the dynamical phase $\phi$ (similarly to the far-field approximation in optics). Inserting Eq. (2.12) into Eq. (3.6), we obtain

$$P_{\text{eff}} \approx 2P_{\text{LZ}} (1 - P_{\text{LZ}}) \left[ 1 - \sum_v D(v) \cos \left\{ \frac{(4\epsilon^{3/2}/3 + 2\epsilon^{1/2}v)}{\hbar^2 \epsilon^{1/2}} - 2\theta_r \right\} \right]$$

$$= 2P_{\text{LZ}} (1 - P_{\text{LZ}}) \times \left[ 1 - \cos \phi_0 \sum_v D(v) \cos \left\{ \frac{t_v}{\hbar} \right\} + \sin \phi_0 \sum_v D(v) \sin \left\{ \frac{t_v}{\hbar} \right\} \right], \quad (3.7)$$
where
\[ \phi_0 = \frac{4e^{3/2}}{3\hbar x^{1/2}} - 2\theta_r, \quad t_s = 2\sqrt{e/\pi}. \]

Note that the two sums in the latter expression for \( P_{\text{eff}} \) are the cosine and sine Fourier transforms of \( D(v) \), with an argument which is proportional to the separation time \( t_s \). Assuming further that the number of degrees of freedom in the bath is large, and that the distribution of \( V_k \)'s satisfies the central limit theorem, then \( D(v) \) is approximated by
\[ D(v) \approx \frac{1}{\sqrt{2\pi v_0}} \exp \left\{ -\frac{v^2}{2v_0^2} \right\}. \]  \hfill (3.8)

In this case, the last sum in Eq. (3.7) vanishes due to symmetry and we obtain
\[ P_{\text{eff}} \approx 2P_{LZ}(1 - P_{LZ}) \left[ 1 - F_A \cos \left\{ \frac{4e^{3/2}}{3\hbar x^{1/2}} - 2\theta_r \right\} \right], \]  \hfill (3.9)
\[ F_A = \exp \left\{ -\frac{t_s^2}{\tau_0^2} \right\}, \]

where \( \tau_0 \sim 1/v_0 \) (\( v_0 \) is the typical coupling strength of a single degree of freedom). The dependence on \( t_s \) of the attenuation factor, \( F_A \), which multiplies the interference term, could be expected in view of the work of Stern et al. [30]. Their work implies that the exponential attenuation of a general interference between two partial waves, for a non-dynamical environment, is quadratic in the time over which a phase difference was accumulated.

In the simplest case where \( V_k = V, \ a_k(-\infty) = a_0, \) and \( b_k(-\infty) = b_0 \) for all \( k \), and for \( a_0 \) not too close to 0 or 1, we obtain
\[ \tau_\phi = \frac{\hbar}{V(2N\min\{|a_0|^2, |b_0|^2\})^{1/2}} \]  \hfill (3.10)
(see Appendix C for details). From Eq. (3.10) it is apparent that given \( V \) and \( N \), \( \tau_\phi \) is minimized for the initial conditions \( a_0 = b_0 = 1/\sqrt{2} \)—i.e., if the environment is prepared in a state which has a minimal overlap with any of the eigenstates of \( H_{\text{en}} \). One can parametrize the initial conditions of the environment as
\[ |a_0|^2 = \frac{e^{-\beta \zeta}}{2 \cosh(\beta \zeta)}, \quad |b_0|^2 = \frac{e^{\beta \zeta}}{2 \cosh(\beta \zeta)}, \]  \hfill (3.11)
so that the dephasing time \( \tau_\phi \) is a monotonically increasing function of the parameter \( \beta \).

Note that so far we have considered the environment to be in a pure quantum state. Alternatively, the calculation of \( P_{\text{eff}} \) can be repeated assuming that the system of interest (described by \( H_0(t) \)) is coupled to a thermal bath; namely, that the
environment is prepared in a mixed state. In that case, \(|a_0|^2\) and \(|b_0|^2\) in Eq. (3.11) are the Boltzman probabilities for occupation of the lower and upper level, respectively, of each degree of freedom in the environment; the parameter \(\beta\) then assumes the physical significance of \(1/k_B T\), where \(T\) is the temperature of the thermal bath. The resulting expression for \(P_{\text{eff}}\) in the presence of a thermal bath is identical to the result obtained for the quantum coherent environment. This equivalence of thermal and coherent states is characteristic to a non-dynamical environment. The reason is that due to the fact that there are no transitions among different states in the environment, the occupation probabilities of these states (the diagonal elements of the corresponding density matrix) remain unchanged. From the general expression for \(P_{\text{eff}}\) in Eq. (3.5), it is obvious that the existence of non-vanishing off-diagonal density matrix elements does not affect the transition probability. In view of the interpretation of \(\beta\) for a thermal bath, Eqs. (3.10) and (3.11) imply that \(\tau_\phi\) decreases (and consequently the dephasing becomes stronger) as function of temperature.

It is interesting to investigate also the particular case, where \(D(v)\) in Eq. (3.6) does not satisfy the central limit theorem. One example is a power-law distribution of the coupling coefficients \(V_k\):

\[
p(V) \sim |V|^\gamma.
\]  
(3.12)

In Appendix D we calculate \(D(v)\), and the implication on \(P_{\text{eff}}\), of the above distribution. In the calculation we consider the simple case where \(a_k(-\infty) = b_k(-\infty) = 1/\sqrt{2}\) for all \(k\) (which corresponds to infinite temperature if the environment is prepared in a thermal state). We find that for the separable case, \(\tau_\gamma \ll t_s\), \(P_{\text{eff}}\) has a form similar to Eq. (3.9), but the exponential attenuation factor which multiplies the interference term is replaced by a power-law attenuation as a function of \(t_s\):

\[
F_A \sim \left| \frac{\tau_\phi}{t_s} \right|^{N(1+\gamma)}.
\]  
(3.13)

It is possible to modify the distribution in Eq. (3.12), making it normalizable, by introducing an upper cutoff \(V_c\) for \(\gamma > -1\), or a lower cutoff \(V_0\) for \(\gamma < -1\). This implies (see Appendix D for details) that the attenuation factor \(F_A\) decays exponentially for time scales \(t_s\) such that \(t_s \ll h/V_c\), \(t_s \gg h/V_0\) for \(\gamma > -1\), \(\gamma < -1\), respectively; for \(h/V_c \ll t_s\), \(t_s \ll h/V_0\), this attenuation factor is a power-law in \(t_s\).

So far we have discussed the separable case \(\tau_\gamma \ll t_s\), in which a two-wave interference term can be clearly identified in the expression for \(P_{\text{eff}}\). We have shown that in this case the effect of dephasing is to multiply the interference term by an attenuation factor, which decays as a function of the separation time \(t_s\). The form of the decay depends on the microscopic details of the environment. To complete the discussion of a non-dynamical environment, we consider also the non-separable case \(\tau_\gamma \gg t_s\). In this case we were not able to evaluate the trace over the environment degrees of freedom analytically. In Fig. 3 we show numerical results for \(P_{\text{eff}}\)
Fig. 3. The asymptotic transition probability, $P_{\text{eff}}$, for the toy model coupled to a two-level degrees of freedom environment (cf. Eq. (3.1), with $H_{\text{en}}$ and $\hat{O}_{\text{en}}$ given by Eq. (3.3)). $P_{\text{eff}}$ is plotted as function of $\varepsilon/(\hbar^2\alpha)^{1/3}$ for various values of $V/(\hbar^2\alpha)^{1/3}$, where the coupling coefficients $V_k$ in Eq. (3.3) satisfy $V_k = V$ for all $k$. Full line corresponds to $V = 0$; $V/(\hbar^2\alpha)^{1/3} = 0.1$ is denoted by crosses; $V/(\hbar^2\alpha)^{1/3} = 0.5$ is denoted by diamonds. Note that the average amplitude of the oscillations in $P_{\text{eff}}$ is larger for $V/(\hbar^2\alpha)^{1/3} = 0.5$ than $V/(\hbar^2\alpha)^{1/3} = 0.1$, indicating nonmonotonicity of the dephasing time as function of the coupling strength.

(cf. Eq. (3.5)) as a function of the separation parameter $\varepsilon$, for a few values of $V$ (assuming $V_k = V$ for all $k$). As expected, the oscillations in $P_{\text{eff}}$ vs $\varepsilon$ (see $V = 0$ in Fig. 3) are attenuated for finite values of $V$. This attenuation reflects the reduction of interference effects, which, in the absence of the external environment, are manifested by these oscillations. We find that the suppression of the amplitude of the oscillations, in the region where $\varepsilon$ is not too large, is not a monotonic function of the coupling strength $V$. The following heuristic argument may shed some light on this result: since the Zener time, $\tau_z$, is finite, one may argue that the transitions between the two adiabatic levels are smeared over time. We thus consider interference between two wave packets, one which Zener tunnels over the time interval $-t_s/2 - \tau_z/2 < t < -t_s/2 + \tau_z/2$, and the other which tunnels over the interval $t_s/2 - \tau_z/2 < t < t_s/2 + \tau_z/2$. The entire interference pattern is obtained by summing over pairs of partial waves (belonging with the two wave packets alluded to above). The dephasing time, $\tau_\phi$, is the same for all pairs. However, the relative phase factor of such a pair and the related attenuation factor, both depend on the separation time (see, e.g., Eqs. (3.7) and (3.9)); one may attribute uncertainty of order $2\tau_z$ to the latter. As a result, the total transition probability (obtained by summing over the contributions of all pairs of partial waves), turns out to be a highly sensitive function of the parameter $\tau_\phi$, and, in particular, a non-monotonic one.
(b) A Dynamical Environment

We next show that a similar dephasing mechanism is provided by an external bath, whose dynamics when coupled to the system of interest is not trivial. We consider the Hamiltonian $H(t)$ (Eq. (3.1)), with $[H_{en}, \hat{O}_{en}] \neq 0$. Following Leggett et al. [22] we consider an environment which consists of a set of harmonic oscillators; the coupling to the system is assumed to be linear in the coordinates of these oscillators:

$$H_{en} = \sum_x \left( \frac{m_x \omega_x^2}{2} x_x^2 + \frac{p_x^2}{2m_x} \right), \quad \hat{O}_{en} = \sum_x C_x x_x. \quad (3.14)$$

The dynamics of this environment can give rise to energy dissipation on top of dephasing. The resulting effect on the transition probability $P_{eff}$ is, in general, quite involved. In the following analysis we treat the simple sudden limit, $\delta \ll 1$ (cf. Eq. (2.13)), of the dynamics associated with the Hamiltonian $H_0(t)$ (Eq. (2.1)). In this case $H_0(t)$ nearly commutes with $\sigma_z$ for almost all times; hence dissipation of energy due to the type of coupling introduced in Eq. (3.1) is negligible [39]. We then generalize the calculations of Leggett et al. [22] and Dorsey [40] to our system, with the time-dependent bias $(x t^2 - \varepsilon)$. Details are given in Appendix E. We consider an ohmic spectrum of the bath, with a cutoff $\omega_c$; the spectral density satisfies

$$J(\omega) = \frac{\pi}{2} \sum_x \left( \frac{C_x^2}{m_x \omega_x} \right) \delta(\omega - \omega_x) = \eta \omega e^{-\omega/\omega_c}. \quad (3.15)$$

After tracing over the degrees of freedom of the bath, we obtain the expressions for the transition probability,

$$P_{eff} = \left( \frac{A}{2\hbar} \right)^2 \sqrt{\frac{\pi \hbar}{x}} \int_{-\infty}^{\infty} dt \cos \left[ \frac{4}{\pi \hbar} Q_1(\tau) \right] \exp \left[ \frac{-4}{\pi \hbar} Q_2(\tau) \right]$$

$$\times \frac{\cos[\epsilon t/\hbar - \alpha t^3/12\hbar - \pi/4]}{\tau^{1/2}} \quad (3.16a)$$

where

$$Q_1(\tau) = \eta \arctan(\omega_c \tau) \quad (3.16b)$$

and

$$Q_2(\tau) = \eta \left( \ln \sqrt{1 + (\omega_c \tau)^2} + \ln \left[ \frac{\beta \hbar}{\pi \tau} \sinh \left( \frac{\pi \tau}{\beta \hbar} \right) \right] \right) \quad (3.16c)$$

Here $\eta$ is the coupling strength defined in Eq. (3.15), and $\beta = 1/k_B T$, where $T$ is the temperature of the bath. Note that for $\eta = 0$ (i.e., in the absence of interaction with the bath), the above expression for $P_{eff}$ is reduced to a form which contains the
integral representation of the function \( A t^2 \left[-c/(\hbar^2 x)^{1/3}\right] \) and coincides with the bare transition probability in the sudden limit given by Eq. (2.14b). To obtain an explicit expression for \( P_{\text{eff}} \), we make some further assumptions. First, similarly to what has been done for the non-dynamical environment, we assume separability, that is, \( \tau_z \ll t_s \). In addition, we assume a large cutoff frequency and a high enough temperature, so that

\[
\omega_c t_s \gg 1 \quad \text{and} \quad t_s \gg \beta \hbar. \tag{3.17}
\]

Finally, we consider a weak coupling limit

\[
\frac{\gamma}{\epsilon} \ll \beta, \quad \frac{\gamma}{\epsilon} \ll \frac{1}{\hbar \omega_c}, \quad \text{where} \quad \gamma \equiv \frac{2\eta}{\pi \hbar}. \tag{3.18}
\]

Under the above assumptions we obtain (see Appendix E for details)

\[
P_{\text{eff}} \approx 2P_{\text{LZ}}(1 - P_{\text{LZ}}) \left[ 1 - F_A \cos \left( \frac{4\epsilon}{3}\frac{3}{3\hbar x^{1/2} - 2\theta} \right) \right],
\]

\[
F_A = \left( \frac{\pi}{\beta \hbar \omega_c} \right)^{2\gamma} \cos(\gamma \pi) \exp \left( -\frac{t_s}{\tau_\phi} \right), \quad \text{and} \quad \tau_\phi = \frac{\beta \hbar}{2\pi \gamma}. \tag{3.19}
\]

Comparing Eq. (3.19) with Eq. (3.9), we find that the dephasing introduced by coupling to a dynamical environment is manifested in a form similar to that of a non-dynamical environment. In both cases, the interference term is multiplied by an attenuation factor \( F_A \). The dependence of \( F_A \) on the separation time \( t_s \) is, however, different; the argument of the exponential factor of \( F_A \) is linear rather than quadratic in \( t_s \). This simple exponential relaxation of interference is characteristic of many physical situations. The expression for the dephasing time \( \tau_\phi \) in Eq. (3.19) is qualitatively similar to the result for a non-dynamical environment (see Eqs. (3.10) and (3.11)); in both cases it is a decreasing function of the coupling strength and the temperature.

4. RELAXATION

In this section we concentrate on the types of coupling to external degrees of freedom which enables exchange of energy between the system and the environment, and thus gives rise to dissipation. We describe the system + environment by a Hamiltonian whose form is similar to \( H(t) \) in Eq. (3.1), but with the operator \( \sigma_z \) in the coupling term replaced by \( \sigma_x \) or \( \sigma_y \). Such a coupling will give rise to transitions between the energy levels of the system, whose self-Hamiltonian is \( H_0(t) \) (Eq. (2.1)), since for most of the time (except near the narrow gaps) \( H_0(t) \) almost commutes with \( \sigma_z \); hence, \( H_0(t) \) does not commute with the coupling term. We discuss here the effects of two different types of a dissipative environment—a coherent
environment, which is prepared in a pure quantum state, and a thermal bath (a mixed state). For the latter we establish an equation of motion for the density matrix of the system, whose solution yields, in particular, the asymptotic transition probability (denoted above as $P_{\text{eff}}$).

(a) Coherent Environment

In order to study the effect of a coherent environment, we introduce here, as an example, a simple model. We consider a bath consisting of two-level systems, similar to the one introduced in the context of dephasing (Section 3(a)) in the presence of a non-dynamical environment. The coupling operator $\hat{O}_{\text{en}}$ is now different, facilitating exchange of energy between the system and the environment. More specifically, the Hamiltonian is

$$H(t) = H_0(t) - \frac{1}{2} \sum_{k=1}^{N} \zeta_k \sigma_z^{(k)} - \frac{1}{2} \sigma_x \sum_{k=1}^{N} V^{(x)}_k \sigma_x^{(k)}, \quad (4.1)$$

where $H_0(t)$ is given by Eq. (2.1) and $\sigma_z^{(k)}$, $\sigma_x^{(k)}$ are the Pauli matrices of the $k$th degree of freedom in the environment. We solve the Schrödinger equation for the above Hamiltonian and trace over the degrees of freedom of the environment (see Appendix F for details). The calculation is performed in the limits $\delta \ll 1$ and $V^{(x)}_k/(\hbar^2 \alpha)^{1/3} \ll 1$ (the former corresponds to the sudden limit, see Es. (2.13)). As a result we obtain the asymptotic transition probability

$$P_{\text{eff}} \approx (\pi \delta)^2 \sum_k \prod_{k \in K} |a_k^{(0)}|^2 \prod_{k \notin K} |b_k^{(0)}|^2 \times \left| \text{Ai}[-\tilde{\varepsilon}] - \left( \sum_{k \in K} g_k \text{Ai}[-\tilde{\varepsilon} + \zeta_k] + \sum_{k \notin K} h_k \text{Ai}[-\tilde{\varepsilon} - \zeta_k] \right) \right|^2, \quad (4.2)$$

where

$$g_k = \left( \frac{V^{(x)}_k}{A} \right) \left( \frac{b^{(0)}_k}{a^{(0)}_k} \right), \quad h_k = \left( \frac{V^{(x)}_k}{A} \right) \left( \frac{a^{(0)}_k}{b^{(0)}_k} \right),$$

$$a_k^{(0)} \equiv a_k(-\infty), \quad b_k^{(0)} \equiv b_k(-\infty), \quad \tilde{\varepsilon} \equiv \frac{\varepsilon}{(\hbar^2 \alpha)^{1/3}}, \quad \zeta_k \equiv \frac{\zeta_k}{(\hbar^2 \alpha)^{1/3}};$$

here the wave function of the system + environment is defined as in Eq. (3.4). We note that the transition probability is defined here in the same way as in Section 2—i.e., $P_{\text{eff}} = |C_+(\infty)|^2$ assuming that the system was prepared with $C_-(\infty) = 1$ and $C_+(\infty) = 0$. If the system is prepared in the upper rather than the lower adiabatic level, that is, with $C_-(\infty) = 0$ and $C_+(\infty) = 1$, the expression for the transition probability $|C_-(\infty)|^2$ is obtained from Eq. (4.2) by exchanging $\zeta_k$ with $-\zeta_k$ in the arguments of the Airy's functions. In order to see the outcome of exchanging $C_-(t)$ and $C_+(t)$ more transparently, we consider Eq. (4.2)
in the simple case where $V_k = V$, $\zeta_k = \zeta$, $a_k(0) = a_0$ (a0 is real and $a_0 \leq 1/\sqrt{2}$ for $\zeta > 0$) and $b_k(0) = \sqrt{1 - |a_0|^2}$ for all k (see Appendix F for details):

$$P_{\text{eff}} = (\pi\delta)^2 (F_s + F_{ns}),$$

where

$$F_s = \left\{ A[i[-\tilde{e} - (NV) a_0 \sqrt{1 - |a_0|^2}(A[i[-(\tilde{e} + \tilde{\zeta})] + A[i[-(\tilde{e} - \tilde{\zeta})])]\right^2$$

and

$$F_{ns} = \left(\frac{NV^2}{\mathcal{A}^2}\right)\left\{a_0^2 A[i[-(\tilde{e} + \tilde{\zeta})] - (1 - |a_0|^2) A[i[-(\tilde{e} - \tilde{\zeta})]\right\}^2.$$ 

The first term in Eq. (4.3), denoted as $F_s$, is symmetric with respect to $\zeta \rightarrow -\zeta$, and consequently to $C_-(t) \rightarrow C_+(t)$. In addition, the effect of the environment with fixed $V$, $\zeta$, and $N$ on $F_s$ is maximal for $a_0 = 1/\sqrt{2}$—i.e., when the environment is prepared in a state which has a minimal overlap with the eigenstates of its Hamiltonian, $H_{en}$. It follows that $F_s$ is not a dissipative term; it shows no preference of relaxation from the upper energy level to the lower over the opposite process. Rather one can interpret the effect of the environment on $F_s$ as fluctuations in the value of the gap, $\mathcal{A}$. In the adiabatic limit, $\delta \gg 1$ (which is the opposite of the limit considered in Eqs. (4.2) and (4.3)), such an effect can be shown to introduce dephasing in the transition regions near the narrow gaps [41]. The second term in Eq. (4.3), denoted by $F_{ns}$, is asymmetric with respect to $\zeta \rightarrow -\zeta$. Hence, this term is dissipative: due to its contribution, the transition probability from \(\rightarrow\) to \(+\rightarrow\) differs from the transition probability from \(+\rightarrow\) to \(\rightarrow\) (the latter is obtained from Eq. (4.3) by reversing the sign of $\zeta$). In particular, for $\zeta \gg \varepsilon$, the transition $\rightarrow \rightarrow$ is less probable than the transition $\rightarrow \rightarrow$ (the latter is obtained from Eq. (4.3) by reversing the sign of $\zeta$). To understand this result, we note that the transitions $|\pm\rangle \rightarrow |\mp\rangle$ are assisted by “emission” or “absorption” of environmental quanta, whose energy $\zeta$ must balance the energy difference between the final and initial states of the system. Hence, for $\zeta \gg \varepsilon$ these transitions occur at the time intervals $t \ll -t_s/2$ or $t \gg t_s/2$, where $|\mp\rangle$ is higher in energy than $|\rightarrow\rangle$ (see Fig. 1). For $\zeta < \varepsilon$, however, the transitions may also take place in the interval $-t_s/2 < t < t_s/2$; during this interval, the state $|\rightarrow\rangle$ is lower in energy than $|\rightarrow\rangle$. Note also that the dissipative term $F_{ns}$, as opposed to $F_s$, is maximal for $a_0 = 0$, and minimal for $a_0 = 1/\sqrt{2}$. This means that the dissipation is more efficient when the difference in occupation probability between the lower and upper levels of the environment is larger.

(b) Thermal Bath

In the previous subsection, we have assumed the environment to be in a coherent (pure) state. For certain classes of environment spectra and coupling, we were able to derive an explicit expression for the asymptotic transition probability. The time evolution in that case is underlined by energy exchange between the system and the
environment (relaxation), but a simple equation of motion (for, e.g., the density matrix) cannot be derived. We therefore discuss now a different scenario, namely, an environment in a thermal (mixed) state. This facilitates a derivation of a simple equation of motion for the density matrix. In particular, when the spectrum of the environment is broad, we obtain a friction term in the equation of motion.

We consider the Hamiltonian

$$H(t) = H_0(t) + H_B + F^\dagger \sigma^- + F \sigma^+, \quad (4.4)$$

where

$$H_B = \sum_k \zeta_k a_k^\dagger a_k \quad \text{and} \quad F = \sum_k V_k a_k.$$

Here $H_0(t)$ is given by Eq. (2.1), $\sigma_\pm = (\sigma_z \pm i \sigma_x)/2$, and $a_k, a_k^\dagger$ are, respectively, the annihilation and creation operators for the $k$th degree of freedom in the bath. In a previous work, Nitzan and Silbey [42] have considered the coupling of a bath to a two-level system, whose form is identical to Eq. (4.4) but with a time-independent Hamiltonian of the bare system ($H_0 = \varepsilon \sigma_z$). The Hamiltonian in Eq. (4.4) is similar to Eq. (4.1), but in Eq. (4.4) we assume the rotating wave approximation—namely, we neglect coupling terms of the form $F^\dagger \sigma_-$ and $F \sigma^-$. In addition, the operators $a_k, a_k^\dagger$ represent a general type of degree of freedom (i.e., they can satisfy either fermion or boson commutation relations). We next assume that the bath is in thermal equilibrium at a temperature $T$, with a density matrix $\rho^{eq}_B = e^{-\beta H_B}/\text{Tr}(e^{-\beta H_B})$, $\beta \equiv 1/k_B T$. Similarly to what has been done in Subsection (a), we also assume the sudden limit, $\delta \ll 1$, and weak interaction, so that our results are valid to second order in both $\Delta$ and $V_k$. The weak interaction limit implies, in particular, that the equilibrium state of the bath is virtually unaffected by the system. We consider the expectation value $P(t)$ of the operator $\hat{P} \equiv (\sigma_+ (t) + 1)/2$, obtained after tracing over the degrees of freedom of the bath. Hereafter $\hat{O}(t)$ denotes the time-dependent Heisenberg representation of the operator $\hat{O}$ (for our present discussion the Heisenberg picture is more convenient than the Schrödinger picture). Physically, the quantity $P(t)$ describes the probability of the system to be in the upper diabatic level $|+\rangle$ at time $t$. Assuming that initially the system has been prepared in the lower level $|-\rangle$, then $P(t)$ denotes the transition probability at finite times. Its asymptotic value $P(\infty)$ is the quantity denoted earlier by $P_{\text{eff}}$. In Ref. [42], Nitzan and Silbey derived an effective equation of motion for $P(t)$, which includes a relaxation term. Below we generalize their calculations for the time-dependent Hamiltonian $H_0(t)$.

Imposing the initial conditions $P(t_0) = \langle \sigma_\pm (t_0) \rangle_T = 0$ (here $\langle O \rangle_T$ denotes thermal average of $O, \text{Tr}(\rho^{eq}_B O)$), we derive an equation of motion for the expectation value $P(t)$, starting from the Heisenberg equations of motion for the operators $\hat{P}(t), \sigma_+(t)$, and $\sigma_-(t)$. Details are given in Appendix G. To second order in $\Delta$ and the coupling term, we obtain

$$\dot{P}(t) \approx C_0(t)(1 - 2P(t)) - \frac{1}{\hbar^2} \left[ \int_{t_0}^t \left[ [V(t), [V(\tau), \hat{P}(\tau)]] \right] \right]_T dt, \quad (4.5)$$
where

\[ V(t) = F^\dagger(t) \sigma_- (t_0) \exp \left\{ -i \int_0^t (\alpha t^2 - \varepsilon) \, dt \right\} \frac{1}{\hbar} \]

\[ + F(t) \sigma_+ (t_0) \exp \left\{ i \int_0^t (\alpha t^2 - \varepsilon) \, dt \right\} \frac{1}{\hbar}. \]

Here \( C_0(t) \) is the part which persists in the absence of coupling to the environment, and determines the bare transition probability (for \( t \to \infty \), cf. Eq. (2.14b) and the integral representation of Airy's function):

\[ C_0(t) = \left( \frac{A}{2\hbar} \right)^2 \exp \left\{ \frac{i}{\hbar} \left( \frac{\alpha t^3}{3} - \varepsilon t \right) \right\} \int_0^t \exp \left\{ -i \frac{1}{\hbar} \left( \frac{\alpha \tau^3}{3} - \varepsilon \tau \right) \right\} d\tau + \text{C.C.} \quad (4.6) \]

After some algebra (see Appendix G), and replacing the sum over \( k \) in the definition of \( F \) (Eq. (4.4)) by an integral over the spectrum, we obtain

\[ \hat{P}(t) = -B(t) P(t) + C(t), \quad (4.7a) \]

where

\[ B(t) = \frac{1}{\hbar^2} \exp \left( \frac{i}{\hbar} \left( \frac{\alpha t^3}{3} - \varepsilon t \right) \right) \int_0^t d\tau \exp \left( -i \frac{1}{\hbar} \left( \frac{\alpha \tau^3}{3} - \varepsilon \tau \right) \right) \]

\[ \times \left( \int_{-\infty}^{\infty} d\zeta \, G(\zeta) \exp \left( \frac{i\zeta (\tau - t)}{\hbar} \right) \langle \{ a^\dagger(\zeta), a(\zeta) \} \rangle_{\tau} + \frac{A^2}{2} \right) + \text{C.C.} \quad (4.7b) \]

and

\[ C(t) = \frac{1}{\hbar^2} \exp \left( \frac{i}{\hbar} \left( \frac{\alpha t^3}{3} - \varepsilon t \right) \right) \int_0^t d\tau \exp \left( -i \frac{1}{\hbar} \left( \frac{\alpha \tau^3}{3} - \varepsilon \tau \right) \right) \]

\[ \times \left( \int_{-\infty}^{\infty} d\zeta \, G(\zeta) \exp \left( \frac{i\zeta (\tau - t)}{\hbar} \right) \langle a^\dagger(\zeta) a(\zeta) \rangle_{\tau} + \frac{A^2}{4} \right) + \text{C.C.}; \quad (4.7c) \]

here the curly brackets denote anti-commutator; C.C. is complex conjugate; \( G(\zeta) \) is the power spectrum, defined as

\[ G(\zeta) = |V(\zeta)|^2 \, D(\zeta), \quad (4.8) \]

where \( D(\zeta) \) is the density of states of the bath. The solution of Eq. (4.7a), for \( P(t_0) = 0 \), has the form

\[ P(t) = \exp \left\{ -\int_{t_0}^t B(\tau) \, d\tau \right\} \int_{t_0}^t \exp \left\{ \int_{t_0}^s B(s) \, ds \right\} C(\tau) \, d\tau. \quad (4.9) \]
We note that for a coherent environment (Subsection (a)) there is no simple equation for $P(t)$ such as Eq. (4.7a). In that case, the same procedure will result in a set of coupled differential equations, which mix $P(t)$ with other quantities (associated with off-diagonal elements in the density matrix); we were not able to treat these analytically.

The function $B(t)$ in Eq. (4.9) describes the rate of the exponential relaxation of $P(t)$ to a steady state ($\dot{P}(t) = 0$). To find the explicit time dependence of this relaxation process, we consider Eqs. (4.7b) and (4.7c) for various types of degrees of freedom and various spectra. In particular, for the case of a singular spectrum ($D(\xi) \sim \delta(\xi - \xi_0)$), $\int_0^t B(\tau) \, d\tau$ is a bounded function of time. Similarly to Eq. (4.3), we then obtain a bounded correction to the asymptotic transition probability $P(\infty)$, such that a transition from an upper to lower level is preferable over the opposite process. In order to obtain a genuine relaxation of $P(t)$, the spectrum of the bath must be sufficiently broad such that $\int_0^t B(\tau) \, d\tau$ is an increasing function of time. For a general choice of $G(\xi)$, we obtain simple approximate expressions for $B(t)$ and $C(t)$ (Eq. (4.7)), provided that the correlation time $\tau_{\text{cor}}$ in the bath is short: we define $\tau_{\text{cor}}$ as the width of the correlation function $K(t)$ (the Fourier transform of $G(\xi)$) and require

\[
\tau_{\text{cor}} \ll \left( \frac{\hbar}{\omega} \right)^{1/3}.
\] (4.10)

This condition is trivially satisfied in the particular case of an ohmic spectrum, where $G(\xi)$ is a constant and hence $K(t) \propto \delta(t - t_0)$ (note: the spectral density $J(\omega)$ introduced by Leggett and Caldeira [26] is related to $G(\xi)$ through $J(\omega) \propto \omega G(\hbar \omega)$). For any value of $\tau_{\text{cor}}$ which satisfies Eq. (4.10), we calculate $B(t)$ and $C(t)$ with $a(\xi), a^\dagger(\xi)$ representing different types of baths (see Appendix G for details). For a bath of fermions we find

\[
B(t) \approx \frac{1}{\hbar^2} \left( G(\pi t^2 - \varepsilon) + 2C_0(t) \right),
\]

\[
C(t) \approx \frac{1}{\hbar^2} \left( G(\pi t^2 - \varepsilon) f(\pi t^2 - \varepsilon) + C_0(t) \right),
\] (4.11)

where $f(E)$ is the equilibrium Fermi–Dirac distribution, $f(E) \equiv 1/(1 + e^{\beta E})$; $C_0(t)$ is defined in Eq. (4.6). For a bath of bosons we obtain

\[
B(t) \approx \frac{1}{\hbar^2} \left( G(\pi t^2 - \varepsilon) \coth \left[ \frac{\beta(\pi t^2 - \varepsilon)}{2} \right] + 2C_0(t) \right),
\]

\[
C(t) \approx \frac{1}{\hbar^2} \left( G(\pi t^2 - \varepsilon) \left( g(\pi t^2 - \varepsilon) + C_0(t) \right) \right),
\] (4.12)
where \( g(E) = 1/(e^{\beta E} - 1) \) is the Bose–Einstein distribution. The general solution for \( P(t) \), obtained by substituting Eq. (4.11) or (4.12) into Eq. (4.9), has a very complicated time-dependence. In the high temperature limit (\( \beta \to 0 \)), where the bath degrees of freedom can be regarded as classical, \( B(t) \) and \( C(t) \) in Eqs. (4.11), (4.12) satisfy

\[
C(t) \approx B(t)/2. \tag{4.13}
\]

The explicit solution for \( P(t) \) is then

\[
P(t) = \frac{1}{2} \left( 1 - \exp \left\{ - \int_{t_0}^{t'} B(\tau) \, d\tau \right\} \right). \tag{4.14}
\]

We next concentrate on the physical interpretation of the functions \( B(t) \) and \( C(t) \). From Eq. (4.7a) one can see that a steady state solution for \( P(t) \) (\( \dot{P}(t) = 0 \)) should satisfy \( P(t) = C(t)/B(t) \). In the presence of a bath of either fermions or bosons (Eqs. (4.11) and (4.12)), this means that the steady state is determined by an interplay between the Zener transitions (described by \( C_0(t) \)) and thermal equilibration with the bath (expressed by function \( f(xt^2 - \varepsilon) \)). We have already mentioned above that the function \( B(t) \) has the physical significance of the rate of exponential relaxation to the steady state. The contribution of the bath to the relaxation rate is

\[
B_{eq}(t) = B(t) - 2C_0(t). \tag{4.15}
\]

In a regime where \( B_{eq}(t) \) is almost a constant, one can define the characteristic time-scale for dissipation as \( \tau_{eq} \equiv 1/B_{eq} \). Similarly to the dephasing time \( \tau_\phi \) (see previous section), \( \tau_{eq} \) is a decreasing function of the coupling strength (contained in the definition of \( G(\zeta) \), see Eq. (4.8)). The dependence of \( \tau_{eq} \) on temperature is different for the various types of baths. In particular, for a bath of fermions it is independent of temperature. This implies a major difference between the characteristic time-scales \( \tau_{eq} \) and \( \tau_\phi \); the latter is a decreasing function of temperature for any type of external environment; i.e., higher temperature always leads to stronger phase uncertainty.

To demonstrate the dependence of the relaxation rate on the spectrum of the bath, we consider a fermion bath with a cutoff in the spectrum

\[
G(\zeta) = G \exp \left\{ - \frac{\zeta}{\zeta_c} \right\}, \tag{4.16}
\]

which is essentially an ohmic spectrum for \( \zeta \ll \zeta_c \). Inserting Eq. (4.16) into \( B(t) \) in Eq. (4.11), and employing Eq. (4.15), we find the exponent of relaxation

\[
\int_{t_0}^{t'} B_{eq}(\tau) \, d\tau = \frac{G}{2\hbar^2} \sqrt{\frac{\zeta}{\alpha}} \exp \left\{ \frac{\alpha}{\zeta_c} \right\} \left( \text{erf} \left[ \sqrt{\frac{\alpha}{\zeta_c}} t \right] - \text{erf} \left[ \sqrt{\frac{\alpha}{\zeta_c}} t_0 \right] \right). \tag{4.17}
\]
For short time-scales, such that \( |t| \ll \sqrt{\zeta_c/\alpha} \), \( |t_0| \ll \sqrt{\zeta_c/\alpha} \), the exponent is thus linear in time:

\[
\int_{t_0}^{t'} B_{eq}(\tau) \, d\tau = \frac{(t - t_0)}{\tau_{eq}} \quad , \quad \tau_{eq} \equiv \frac{G}{\hbar^2} .
\] (4.18)

For longer time-scales \( t \gg \sqrt{\zeta_c/\alpha} \) and \( t_0 \ll -\sqrt{\zeta_c/\alpha} \), the exponent saturates and approaches a constant:

\[
\int_{t_0}^{t'} B_{eq}(\tau) \, d\tau \to \frac{G}{2\hbar^2} \sqrt{\frac{\zeta_c}{\alpha}} \exp \left\{ \frac{\epsilon}{\zeta_c} \right\} .
\] (4.19)

The saturation time \( \sqrt{\zeta_c/\alpha} \) corresponds to the time in which the energy difference between the levels of the system exceeds the cutoff in the spectrum of the environment. Then, obviously, there are no channels for dissipation of energy from the system to the bath. (We have neglected in our model "multiphonon processes."

Finally, we note that one can derive an equation of motion, similar to Eq. (4.7a), for all elements of the two-level density matrix \( \rho(t) \) is a diagonal matrix element). The derivation is sketched in Appendix G. We find that the relaxation rate of the off-diagonal matrix elements is slower by a factor of 2 than the relaxation rate of the diagonal elements. The relaxation rate of the off-diagonal elements has, in general, an imaginary part. The latter vanishes if \( G(\zeta) \) is a symmetric (anti-symmetric) function of \( \zeta \), for a bath of fermions (bosons). In this case the equations of motion are of the form

\[
\dot{\rho}_{mn}(t) = -B(t) \rho_{mn}(t) + C_{mn}(t) ,
\]

\[
\dot{\rho}_{nn'}(t) = -\frac{B(t)}{2} \rho_{nn'}(t) + C_{nn'}(t) \quad (n \neq n') ;
\] (4.20)

here \( \rho_{nn'}(t) \) are the elements of the density matrix \( \rho(t) \). In the next section we assume that a similar equation applies for any quantum system (in particular, of a larger number of energy levels), in the presence of a dissipative environment.

5. THE CONDUCTANCE OF A MULTI-LEVEL SYSTEM

In the previous sections we have discussed various effects of an external environment on a quantum mechanical system. We have demonstrated the onset of dephasing and dissipation. In particular, we have shown that in the absence of dissipation, interreference terms decay exponentially (in the generic case) with a time-scale \( \tau_\phi \). Formally, this dephasing can be introduced in the equation of motion for the density matrix as a relaxation term of the off-diagonal elements only. When dissipation is present, we have shown that the off-diagonal elements of the density
matrix decay exponentially (in the generic case) with a time-scale $2\tau_{eq}$. In addition, the diagonal elements decay towards their equilibrium values at a rate $1/\tau_{eq}$. In case of a general coupling, the two effects are assumed to be additive; the effective dephasing rate, $1/\tau_\phi$ (when both dissipative and non-dissipative couplings are present) is given by

$$\frac{1}{\tau_\phi} = \frac{1}{2\tau_{eq}} + \frac{1}{\tau_\phi}. \quad (5.1)$$

The characteristic relaxation time, $\tau_{eq}$, is typically different from $\tau_\phi$. We now incorporate both effects of the external bath in the dynamics of a more complicated quantum system; we consider a multi-level system in the presence of an external driving source. The interplay between Zener dynamics, dephasing, and dissipation gives rise to a highly nonlinear behavior of the response to the driving source.

Below we consider the model discussed earlier by Gefen and Thouless [14], to which we add phenomenologically both dephasing and dissipation (originally induced by an external environment). The model, in the absence of coupling to the environment, describes, e.g., an electron in a one-dimensional ring, subject to a constant e.m.f. $V$ induced by a time-dependent magnetic flux $\phi(t) = cVt$. The corresponding Hamiltonian is

$$H_0(t) = \frac{\hbar^2}{2M} \left(-i \frac{\partial}{\partial x} - \frac{\phi(t)}{L\phi_0} \right)^2 + U(x); \quad (5.2)$$

![Fig. 4. The instantaneous (adiabatic) spectrum of Eq. (5.2) (schematic), as function of time, for $\phi(t) = cVt$. The spectrum is a periodic function of time, with a period $\tau_0$. The times $t_i$, $t_m$, and $t_f$ are referred to in the text (Section 5).](image)
here $x$ is the coordinate along the ring, $\phi_0$ is the flux quantum, $L$ is the circumference of the ring, and $U(x)$ is some weak scattering potential. The instantaneous (adiabatic) spectrum associated with the above Hamiltonian is a periodic function of time with a period $\tau_0 = 2\pi\hbar/eV$ (see Fig. 4). We assume that the dynamics of an electron in the system is dominated by Landau–Zener transitions, which take place only in the vicinity of the narrow gaps between adjacent levels. We also assume that the time of Zener tunneling at each transition region is much smaller than the period of the spectrum, so that during each half-period between consecutive narrow gaps (e.g., the intervals $t_i^+ < t < t_i^-$ or $t_m^+ = t < t_m^-$ in Fig. 4) the system follows the instantaneous energy levels adiabatically. For simplicity we assume that $U(x)$ is such that all the narrow gaps are equal, and the transition amplitudes in their vicinity are essentially independent of the level index. This assumption is reasonable since we are interested in the dynamics of electrons in the close vicinity of the Fermi level. Formally, recalling the periodicity of the spectrum, we approximate the time-evolution of the electron density matrix $\rho(t)$ (in the absence of coupling to the environment), by the following evolution equation: for $t = m\tau_0/2$, $m$ integer,

$$\rho(t + \tau_z/2) = S\rho(t - \tau_z/2) S^\dagger,$$

(5.3a)

$$\rho(t + \tau_0/2 - \tau_z/2) = A\rho(t + \tau_z/2) A^\dagger.$$

(5.3b)

Here we represent $\rho(t)$, $S$, and $A$ by matrices, whose indices coincide with the level indices. The Zener time, $\tau_z$, is assumed to be vanishingly small. $S$ is a scattering matrix, and is decoupled into $2 \times 2$ blocks $S(n, n + 1)$, each having the form of the scattering matrix $S$ defined in Eq. (2.7); $S(n, n + 1)$ describes the dynamics associated with transitions between the $n$th and the $(n + 1)$th levels. The parity of $n$ in $S(n, n + 1)$ is identical to the parity of $m$, and the reflection and transmission amplitudes $\tilde{r}$ and $\tilde{t}$ (see Eqs. (2.6) through (2.8)) are assumed here to be the same for all blocks, with $x_{\text{eff}}$ proportional to $V$. $A$ is a diagonal matrix, which represents the phase factor accumulated by off-diagonal elements of $\rho$ between consecutive transitions,

$$A_{nn} = \exp\{-iE_n\tau_0/2\hbar\},$$

(5.4)

where $E_n$ is the average energy of the $n$th adiabatic level. As was noted above, focusing on the dynamics of electrons near the Fermi level, the level-spacing $\Delta E$ is approximately constant, hence $(E_n - E_F) \approx n \Delta E$; hereafter $n$ is measured relative to the Fermi level index $n_F$.

We next assume that the system is coupled to an external environment, giving rise to dissipation. Following the analysis of Section 4, we represent this coupling as a relaxation term in the equation of motion for the density matrix $\rho(t)$. We assume that Eq. (4.20), which has been derived for a two-level system, can be generalized to a multi-level system. Note that Eq. (4.20) was obtained assuming that the environment is prepared in a thermal (mixed) state. Provided that the spectrum is broad (cf. Section 4 for details) it may be possible to approximate the
relaxation rate $B_{eq}(t)$ (cf. Eqs. (4.15) and (4.20)) by a time independent constant, $1/\tau_{eq}$. In this case, one may also rewrite $C_{nn}(t)$ in Eq. (4.20) as

$$C_{nn}(t) = C_{nn}^{(0)}(t) + \frac{\rho_{nn}^{eq}(t)}{\tau_{eq}},$$

(5.5)

where $\rho_{nn}^{eq}(t)$ are the diagonal elements of the density matrix at thermal equilibrium with the bath, and $C_{nn}^{(0)}$ is determined by the Zener dynamics (cf. Eqs. (4.11) and (4.12)). The free terms in the equation of motion for the off-diagonal elements ($C_{nn'}(t)$ with $n \neq n'$, cf. Eq. (4.20)), are solely determined by the Zener dynamics. Following the above assumptions, we cast Eq. (4.20) (extended to the multi-level model at hand) in the form

$$\dot{\rho}_{nn}(t) = -\frac{i}{\hbar} \left[H_0(t), \rho(t)\right]_{nn} - \frac{(\rho_{nn}(t) - \rho_{nn}^{eq}(t))}{2\tau_{eq}},$$

$$\dot{\rho}_{nn'}(t) = -\frac{i}{\hbar} \left[H_0(t), \rho(t)\right]_{nn'} - \frac{\rho_{nn'}(t)}{2\tau_{eq}}, \quad n \neq n'.$$

(5.6)

Equation (5.6) is obtained by substituting Eqs. (4.15) and (5.5) into Eq. (4.20), and identifying all terms which are independent of the bath as giving rise to the bare Zener dynamics, i.e., the first term on the right-hand side of Eqs. (5.6). For relaxation times $\tau_{eq}$ large compared with the period $\tau_0$, one may neglect the time-dependence of $\rho_{nn}^{eq}(t)$, and replace it by the equilibrium distribution for the time averaged energy levels $E_n$. For electrons (or holes) excited above the Fermi level at low temperatures, $k_B T \ll \Delta E$, one may approximate $\rho_{nn}^{eq}$ by the Boltzman distribution

$$\rho_{nn}^{eq} = \frac{e^{-\zeta n}}{\text{Tr}(e^{-\zeta n})}, \quad \zeta = \frac{\Delta E}{k_B T},$$

(5.7)

The dissipative term in Eq. (5.6) leads to a simple exponential relaxation of $\rho(t)$ towards $\rho^{eq}$. To include the effect of an additional, non-dissipative coupling as well, we introduce an additional exponential attenuation of the off-diagonal elements of $\rho(t)$ only, with a rate $1/\tau_{\phi}$. We note that in the previous sections we have related this phenomenological description to microscopic models, and in particular we have shown that this simple exponential decay is generic. As a result of the inclusion of dephasing and dissipation, the time-evolution of $\rho(t)$ between the narrow gaps is modified. We neglect here the effect of the bath on a single Zener transition [23, 24]. During each half a period $m\tau_0/2 < t < (m + 1)\tau_0/2$, we replace Eq. (5.3b) by

$$\rho_{nn}(t) = \rho_{nn}^{eq} + \exp \left\{ - \frac{(t - m\tau_0/2)}{\tau_{eq}} \right\} \left( \rho_{nn}\left(\frac{m\tau_0}{2} + \frac{\tau_{\phi}}{2}\right) - \rho_{nn}^{eq}\right),$$

$$\rho_{nn'}(t) = \exp \left\{ -\left(\frac{i(E_n - E_{n'})}{\hbar} + \frac{1}{2\tau_{eq}} + \frac{1}{\tau_{\phi}}\right)(t - m\tau_0/2) \right\} \rho_{n'}\left(\frac{m\tau_0}{2} + \frac{\tau_{\phi}}{2}\right), \quad n \neq n'.$$

(5.8)
Fig. 5. The expectation value of the energy $\langle E(t) \rangle / E_0$ as a function of time $(t/\tau_0)$ for various values of the e.m.f., $V$. Here $E_0 = 1 \mu eV$, while $\tau_0 = 2\pi \hbar / eV$ is the period of the adiabatic spectrum associated with Hamiltonian (5.2). We distinguish three different values of the dephasing time, $\tau_0$: (a) $E_0 \tau_0 / \hbar = 10^{-2}$ (strong dephasing); (b) $E_0 \tau_0 / \hbar = 40.0$; and (c) $E_0 \tau_0 / \hbar = 10^3$ (which corresponds to $\tau_0 = 2\tau_{eq}$, cf. Eq. (5.1)). All three figures were obtained for $E_0 \tau_{eq} / \hbar = 20.0$, $E_0 \tau_\zeta / \hbar = 1.0$, $\Delta E / E_0 = 100$, and $\zeta = 10$ (cf. Section 5). Following a transient behavior, the system approaches a steady state where the average energy saturates. Note that in Figs. (b) and (c), which correspond to weak dephasing ($\tau_0 \gg \tau_0$), the steady state energy is not a monotonous function of $V$, due to quantum fluctuations.
We note that dephasing is present in the system even for \( \tau_\phi \to \infty \), since then \( \tau_\alpha = 2 \tau_{eq} \) (cf. Eq. (5.1)).

Employing Eqs. (5.3a) and (5.8), we study numerically the time-evolution of \( \rho(t) \) and related physical quantities. In particular, we calculate the expectation value of the energy

\[
E(t) = \sum_n E_n \rho_{nn}(t).
\]

In Fig. 5 we plot \( E(t) \) as function of time, for various values of the e.m.f. \( V \) and the parameters \( \tau_\phi, \tau_{eq} \). We find that in all cases, the energy \( E(t) \) increases on the average over a certain finite transient time. Following this transient, the system approaches a steady state, in which the density matrix (and consequently all physical quantities) averaged over a period is constant. This steady state is achieved when the energy dissipated into the external bath within one period exactly balances the energy pumped into the system by the driving source. For finite values of \( \tau_\phi \), we observe quantum fluctuations in \( E(t) \) which decay during the transition and completely disappear when the steady state is approached [43]. These fluctuations originate from interference effects, and hence are suppressed more rapidly as \( \tau_\phi \) decreases. In particular, in the limit \( \tau_\phi \ll \tau_0 \) (Fig. 5a) the quantum fluctuations will not show up even during the transient time. In the limit \( \tau_\phi \gg \tau_{eq} \) (Fig. 5c), the relevant time-scale which characterizes the decay of interference is \( 2 \tau_{eq} \), since (as noted above) it describes the rate of dephasing which accompanies thermalization with the bath. We note also that for \( \tau_\phi \gg \tau_0 \) (Figs. 5b and c) the steady state energy exhibits fluctuations as a function of \( V \).

Following the above picture, we note that one may relate the energy \( \text{dissipated} \) within one period (in the steady state), \( (\Delta E)_{\text{dis}} \), to the current in the ring, \( I \). In the steady state

\[
(\Delta E)_{\text{dis}} = (\Delta E)_{\text{pumped}},
\]

where \( (\Delta E)_{\text{pumped}} \) is the energy which is being pumped into the system over one period, due to Zener transitions. We may write

\[
I = -c \frac{(\Delta E)_{\text{pumped}}}{\phi_0} = -\frac{1}{V} \frac{(\Delta E)_{\text{pumped}}}{\tau_0}.
\]

From Eqs. (5.10) and (5.11),

\[
\frac{(\Delta E)_{\text{pumped}}}{\tau_0} = \sum_n E_n \left( \rho_{nn}^{ss} - \rho_{nn}^{eq} \right),
\]

where \( \rho^{ss} \) is the steady state solution of the master equation for the density matrix. Note that Eqs. (5.11) and (5.12) yield the current carried by a single electron. Below
we calculate $\rho^{ss}$ analytically for a few simple cases, and express it as a function of $V$. As a result we obtain the single electron conductance

$$G(V) \equiv \frac{I(V)}{V}. \quad (5.13)$$

We first consider the limit $\tau_{\phi} \ll \tau_0$, in which off-diagonal elements of the density matrix are negligible. In Appendix H we show that if the Zener transition probability near the narrow gaps is much smaller than unity, and assuming $\tau_{eq} \gg \tau_0$, then the dynamics of $\rho(t)$ is determined to a good approximation by a diffusion equation with a relaxation term:

$$\dot{\rho}_n(t) = D(V)(\rho_{n+1}(t) - 2\rho_n(t) + \rho_{n-1}(t)) - \frac{(\rho_n(t) - \rho_n^{eq})}{\tau_{eq}}, \quad (5.14a)$$

with

$$D(V) \equiv \frac{e^{-\nu_0/V}}{\tau_0} = \frac{Ve^{-\nu_0/V}}{(2\pi h/e)}. \quad (5.14b)$$

Here $\rho_n(t)$ stands for the diagonal matrix element $\rho_{nn}(t)$, and the exponential factor in $D(V)$ is the Landau–Zener transition probability ($V_0 \equiv E_g^2ML^2/4h^2en_F$, where $E_g$ is the narrow energy gap between the adiabatic levels). We find a solution for the steady state density matrix, $\rho_n^{ss}$, substituting $\dot{\rho}_n(t) = 0$ in Eq. (5.14) (see Appendix H for details):

$$\rho_n^{ss} = \frac{2 \sinh(\zeta/2)}{(1 - f^2(V))} \left\{ e^{-\zeta(n-1/2)} - f(V) e^{-\eta(n-1/2)} \right\}, \quad (5.15)$$

where

$$f(V) \equiv \sqrt{D(V) \tau_{eq}} 2 \sinh(\zeta/2), \quad \eta \equiv 2 \sinh^{-1}(1/2 \sqrt{D(V) \tau_{eq}});$$

$\zeta$ is defined in Eq. (5.7). Inserting the above result for $\rho^{ss}$ into Eqs. (5.11) through (5.13), we obtain an expression for the (single electron) conductance,

$$G(V) = \frac{AE \sqrt{D(V) \tau_{eq}} \sinh(\zeta/2)}{\tau_{eq} V^2 \sinh((\eta + \zeta)/2)}, \quad (5.16)$$

where $D(V)$ and $\eta$ are functions of $V$, given explicitly in Eqs. (5.14b) and (5.15), respectively. In Fig. 6a we plot the numerical result for $G(V)$ as function of $V$, in the limit $\tau_{\phi} \ll \tau_0$, and compare it with the analytic expression (Eq. (5.16)). We note that the analytic expression agrees with the numerical results for small $V$. For larger values of e.m.f. the analytic expression is less satisfactory because our assumption of small transition probability breaks down; consequently, the diffusion coefficient
$D(V)$ is underestimated. We note that for large $V$ the conductance decreases as a power of $V$, whereas for $V \to 0$ its behavior is dominated by the essential singularity in the expression for $D(V)$ (Eq. (5.14b)), the latter being related to the essential singularity in the expression for the Landau–Zener probability. $G(V)$ has a maximum in the crossover from one behavior to the other.

We next consider the regime where $\tau_\phi$ is large and quantum interference effects become significant. As was pointed out in Ref. [14], the dynamics of the system subject to the Hamiltonian $H_0(t)$ (Eq. (5.2)) in the energy space, is analogous to electron localization in one dimension. The corresponding localization length in the energy direction (measured in units of number of levels) is (cf. Ref. [14])

$$
\xi = \frac{2V}{V_0}
$$

(5.17)

(this is within the simple model where all transition probabilities near narrow gaps assume the same value). We define also a localization time $\tau_\xi$, which characterizes the time required for the width of a wave packet to approach $\xi$:

$$
\tau_\xi \equiv \frac{2\pi h \xi}{eV} = \frac{4\pi h}{eV_0}.
$$

(5.18)

Following the analogy with localization in real space, we expect the behavior of the system (averaged over quantum fluctuations) in the presence of dephasing to be dominated by a diffusion process [8], with a diffusion coefficient

$$
D_\xi(V) \sim \frac{\xi^2}{2\tau_\phi}
$$

(5.19)

(cf. Eq. (5.1)). We note again that if $\tau_\phi \gg \tau_\text{eq}$, the dephasing is dominated by the relaxation to thermal equilibrium with the bath. We then assume that the dynamics is described by an equation of the form of Eq. (5.14), with the diffusion coefficient $D(V)$ replaced by $D_\xi(V)$. Using the same procedure as for the case $\tau_\phi \ll \tau_0$, we obtain an expression for the conductance:

$$
G(V) = \frac{\sqrt{2} \Delta E \sinh(\xi/2)}{V_0 \sqrt{\tau_\text{eq} \tau_\phi} V \sinh^{-1}(V_0 \sqrt{\xi/2 \tau_\phi/2 \sqrt{2\tau_\text{eq} V}} + \xi/2)}.
$$

(5.20)

For low temperatures ($\xi \gg 1$), the above expression is considerably simplified, and $G(V)$ in the regime of high $V$ becomes proportional to $1/V$. We note that the picture of a diffusion process characterized by $D_\xi(V)$ makes sense only provided that $\xi > 1$. The case $\xi < 1$ corresponds to a situation where the transition probability from the initial adiabatic level to the next level is exponentially small. We thus expect Eq. (5.20) to breakdown for very small $V$ (from Eq. (5.17)—$V$ and $\xi$ are proportional); in that limit $G(V)$ becomes exponentially small as well. On top of this simple behavior, we expect to observe quantum fluctuations in the conductance. The above expectations are borne out by numerical results, presented in Figs. 6b and c, and compared with the analytic expression (Eq. (5.20)). For $\xi > 1$,
Fig. 6. The single-electron conductance, $G(V)h/e^2$, as a function of the e.m.f., $eV/E_0$, for various values of the dephasing time. The energy scale $E_0$ is defined in the caption of Fig. 5. The values of the parameters $\tau_\phi$, $\tau_\text{eq}$, $\tau_\xi$, $\Delta E$, and $\zeta$ in Figs. (a), (b), and (c) are identical to those of Figs. 5(a), (b), and (c), respectively. Diamonds denote approximate analytic expressions (Eq. (5.16) in Fig. 6a, and Eq. (5.20) in Figs. 6b and c), while full lines represent numerical calculations.

Eq. (5.20) describes $G(V)$ vs $V$ (coarse grained over quantum fluctuations) to a good approximation. This correspondence breaks down for $\xi < 1$, where $G(V)$ becomes identical to the curve obtained for small $V$ in the limit of small $\tau_\phi$ (cf. Fig. 6a).

In the above discussion, $G(V)$ describes the single particle contribution to the conductance. To estimate the total conductance of a metallic ring, we assume that the expectation value of the energy in the many-body state is the sum over single particle contributions, $\langle E \rangle$. We note that this assumption becomes an exact state-
ment in the absence of coupling to the environment [44]. The number of electrons which participate in the conduction is, roughly, \( \langle E \rangle / \Delta E \) (where \( \Delta E \) is the level spacing). Hence, the total energy of the system in the steady state (measured with respect to the ground state energy) is

\[
\langle E_{\text{tot}} \rangle \sim \frac{\langle E \rangle^2}{\Delta E}
\]  
(5.21)

(this is in crude analogy with the finite temperature correction to the mean energy in a Fermi gas, with \( k_B T \) replaced by \( \langle E \rangle \)). Dividing by \( V^2 \tau_0 \) (cf. Eqs. (5.11) and (5.13)), we obtain

\[
G_{\text{tot}}(V) \sim \frac{\langle E \rangle}{\Delta E} \cdot G(V).
\]  
(5.22)

The total conductance \( G_{\text{tot}}(V) \) is plotted vs \( V \) in Figs. 7a–c, for various values of \( \tau_0 \). Similarly to the analysis of the single particle diffusion, employed above to derive expressions for \( G(V) \), we now obtain analytic expressions for \( G_{\text{tot}}(V) \). In the steady state, the single particle incremental energy is

\[
\langle E \rangle = \sum_n E_n \rho_{nn}^{\text{ss}} = \frac{e^{\xi/2} - e^{\eta/2} f(V)}{2 \sinh(\xi/2)(1 - f^2(V))},
\]  
(5.23)

where \( f(V) \) and \( \eta \) are defined in Eq. (5.15), in terms of the diffusion coefficient \( D(V) \). Here \( D(V) \) is given by either (5.14b) for \( \tau_0 \ll \tau_0 \), or (5.19) for \( \tau_0 \gg \tau_0 \). Inserting Eqs. (5.16) and (5.23) into Eq. (5.22), we obtain

\[
G_{\text{tot}}(V) \sim \frac{\Delta E \sqrt{D(V) \tau_{\text{eq}}}}{2 \tau_{\text{eq}} V^2 \sinh(\eta + \xi/2) \{1 - 4D(V) \tau_{\text{eq}} \sinh^2(\xi/2)\}}.
\]  
(5.24)

In the limit \( \xi \gg 1 \) (low temperature), Eq. (5.24) is reduced to a simple expression, which depends crucially on the behavior of \( D(V) \). In the limit \( \tau_0 \ll \tau_0 \) we obtain for small values of \( V \) (in which \( D(V) \) is exponentially small, cf. Eq. (5.14b))

\[
G_{\text{tot}}(V) \sim (\Delta E) D(V),
\]  
(5.25)

which is in good agreement with the numerical results (Fig. 7a). For large \( V \),

\[
G_{\text{tot}}(V) \sim \frac{(\Delta E) D(V)}{V^2}.
\]  
(5.26)

This expression is not compatible with the numerical results in the limit \( \tau_0 \ll \tau_0 \), which show a tendency to level off for large \( V \) (cf. Fig. 7a). As mentioned above, this is due to the fact that in the analytic treatment we underestimate the diffusion coefficient (cf. Appendix H). In the limit \( \tau_0 \gg \tau_0 \), Eq. (5.26) with \( D(V) \) given by
Fig. 7. The total conductance of the ring, $G_{\text{tot}}(V)h/e^2$, as a function of the e.m.f., $eV/E_0$, for various values of the dephasing time (cf. caption of Fig. 5 for definition of $E_0$). The values of the parameters $\tau_\phi$, $\tau_{eq}$, $\tau_\zeta$, $dE$, and $\zeta$ in Figs. 7a, b, and c are identical to the parameters used in Figs. 5 and 6 with the corresponding labels.

Eq. (5.19) is in a good agreement with the numerical results for the conductance (Figs. 7b and c), averaged over quantum fluctuations. This result implies that (provided $\zeta > 1$) $G_{\text{tot}}(V)$ is approximately constant as a function of $V$.

6. DISCUSSION

To complete this paper we summarize our main results, and comment on their relevance to an experimental setup. In the present work we have introduced a
mechanism for the onset of dissipation in mesoscopic systems, driven by external sources. We have studied a toy model (Sections 2 through 4) for a quantum system coupled to various types of external environments. This *microscopic* model is used to derive an effective equation of motion for the density matrix. We have shown that coupling to a thermal bath with a broad spectrum leads to a simple relaxation term in the equation of motion. We have also demonstrated that some types of coupling to the environment lead to attenuation of the off-diagonal elements of the density matrix only; this attenuation is referred to as dephasing. Based on the analysis of these microscopic toy models, we proceed with a phenomenological description of the dynamics of the density matrix in the multi-level, multi-crossing scenario. Within this approach the density matrix relaxes towards thermal equilibrium. The relaxation rate of the off-diagonal elements differs, in general, from the relaxation rate $1/\tau_{eq}$ of the diagonal elements, and no less than $1/2\tau_{eq}$. The dynamics discussed in Section 5 seems relevant for the description of a small 1D metallic ring, threaded by a time-dependent magnetic flux. The onset of dissipation in this system involves two conceptual stages: in the first stage the system absorbs energy from the driving source through Zener dynamics, while in the second this energy is dissipated to the bath. The process of energy pumping into the system is enhanced by dephasing, which leads to a destruction of localization effects. The conductance of the system is calculated in the steady state, where the energy pumped into the system is balanced by the energy dissipated to the bath. We find that for small values of the driving e.m.f. the conductance is dominated by the exponentially small Zener transition probability between consecutive energy levels. At larger values of the e.m.f. the conductance is determined by the interplay between a diffusion process (whose rate depends on both the individual Zener probabilities and the dephasing strength), and a relaxation to thermal equilibrium with the bath. In the regime of weak dephasing, this behavior is modulated by quantum fluctuations. We note that the conductance (averaged over quantum fluctuations) is enhanced by increasing the dephasing rate. On the other hand, it is weakly dependent on the relaxation rate to equilibrium, $\tau_{eq}$. The reason is that decreasing $\tau_{eq}$ leads to a reduction of the steady state energy, but at the same time enhances the rate at which energy is being dissipated to the bath.

As mentioned above, the model introduced in Section 5 is suitable for the description of a metallic loop, which responds to an e.m.f. induced by a linearly time-dependent magnetic flux. For this system we are concerned with single electron levels in a rather limited energy range about the Fermi level. If this energy interval (of the order of $\langle E \rangle$ in the steady state) is small enough, our assumption that the interlevel Zener transition probabilities do not depend systematically on the level index can be justified [45]. The analysis presented in Section 5 may be relevant for an experimental setup, which includes a quasi one-dimensional configuration of an electron gas. Practically it is extremely difficult to produce a metallic ring whose width does not exceed a few Å. One may rather consider, for example, either an annular or a cylindrical geometry of a 2D electron gas, of width $\sim 100\text{Å}$ and a circumference $L \sim 1\text{µm}$. For $E_F \sim 10\text{meV}$ (compatible with an electron density
n_s \sim 10^{12} \text{ cm}^{-2}, and an effective mass equals to 0.1 of the free electron mass), the level spacing is \( \Delta E \sim 100 \text{ \mu eV} \). Such a configuration at temperature \( T = 100 \text{ m}^\circ \text{K} \) satisfies the relation \( \Delta E/k_B T = 10 \), used in our numerical calculations (cf. Section 5). In Figs. 5 through 7 the minimal energy gaps, \( E_g \), between consecutive levels (opened due to disorder) are roughly \( E_g \sim 10 \text{ \mu eV} \) (i.e., 0.1 of the average level spacing). The environment in the experimental system may be provided by, e.g., coupling to phonons. Our numerical results correspond to a characteristic equilibration time \( \tau_{\text{eq}} \sim 10^{-8} \text{s} \). A value of \( \tau_{\text{eq}} \sim 10^{-11} \text{s} \) has been used for the strong dephasing regime. These relaxation times may be controlled by varying the temperature.

The above estimates suggest that the effects discussed in the present work may be observed in realistic systems. We note, however, that our results are based on certain approximations and rough estimates. In particular, we have employed a heuristic argument to evaluate the conductance of a ring, knowing the contribution of a single electron. A more detailed consideration of many-body effects on the dynamics is required and left for future studies. Extending our analysis to a wider range of temperatures is straightforward. Few other open questions awaiting further analysis include a detailed description of the transient behavior, as well as the quantum fluctuations of the conductance. To facilitate comparison with experiments, it may be worthwhile to extend our model to higher dimensional systems and other types (not necessarily linear) of time-dependent bias.

**APPENDIX A: REFLECTION AND TRANSMISSION COEFFICIENTS IN THE LANDAU–ZENER PROBLEM**

Following Zener and Landau [35] we consider a two-level system, subject to the time-dependent Hamiltonian

\[
H(t) = (\pi t \sigma_z + \Delta \sigma_\uparrow)/2.
\]  

(A.1)

We represent the corresponding wave-function in the diabatic basis as

\[
\psi(t) = C_-(t) \exp\{-i\phi(t)\} |-\rangle + C_+(t) \exp\{i\phi(t)\} |+\rangle,
\]  

(A.2)

where \( \sigma_z |\pm\rangle = \pm |\pm\rangle \) and \( \phi(t) = (\pi/2\hbar) \int t' dt' \). We define the transmission and reflection amplitudes \( \tilde{r} \) and \( \tilde{t} \), respectively, as follows: assuming that \( C_-(-\infty) = 0 \) and \( C_+(-\infty) = 1 \), then

\[
\tilde{r} = C_- (\infty) \quad \text{and} \quad \tilde{t} = C_+ (\infty);
\]  

(A.3)

these definitions are compatible with the definition of \( S \) in Eq. (2.7). In order to
calculate \( \tilde{\tau} \) and \( \tilde{\ell} \) we insert \( \psi(t) \) from Eq. (A.2) into the time-dependent Schrödinger equation, and obtain two coupled differential equations for \( C_{\pm}(t) \) (cf. Ref. [35]):

\[
\dot{C}_{\pm}(t) = \frac{i\Delta}{2\hbar} \exp \left\{ \pm \frac{i\alpha t^2}{2\hbar} \right\} C_{\pm}(t).
\]

(A.4)

The amplitudes \( \tilde{\tau} \) and \( \tilde{\ell} \) are then given by the asymptotic solution of Eq. (A.4) with the initial conditions specified above. Below we sketch the solution in two extreme limits, determined by the dimensionless parameter

\[
\gamma = \frac{A^2}{\chi \hbar};
\]

(A.5)

The Sudden Limit, \( \gamma \ll 1 \).

We rewrite Eq. (A.4) in terms of a dimensionless time \( \tau = t \sqrt{\alpha /\hbar} \):

\[
\frac{dC_{\pm}(\tau)}{d\tau} = \frac{i\sqrt{\gamma}}{2} e^{\pm i\gamma \tau^2/2} C_{\pm}(\tau).
\]

We express the solutions of Eq. (A.6) as power series in \( \sqrt{\gamma} \),

\[
C_{\pm}(\tau) = \sum_{n=0}^{\infty} C_{\pm}^{(n)}(\tau) \gamma^{n/2}.
\]

(A.7)

For \( \gamma \ll 1 \), we calculate \( C_{\pm}(\infty) \) to lowest order in \( \sqrt{\gamma} \). The initial conditions imply that \( C_{\pm}^{(0)} = 0 \) and \( C_{\pm}^{(0)} = 1 \). Inserting the above expansion into Eq. (A.6) and integrating over \( \gamma \), we find to lowest order in \( \sqrt{\gamma} \),

\[
C_{-}(\infty) = \frac{i\sqrt{\gamma}}{2} \int_{-\infty}^{\infty} e^{iy^2/2} dy = i\sqrt{\gamma} \int_{0}^{\infty} e^{iy^2/2} dy.
\]

(A.8)

Using the analytic properties of the integrand in the complex plane, one can easily see that the integral on the right-hand side of Eq. (A.8) is equal to the integral along the line \( y = xe^{i\pi/4}, 0 < x < \infty \). We thus obtain

\[
\tilde{\tau} = C_{+}(\infty) = -i\sqrt{\gamma} \int_{0}^{\infty} e^{-x^2/2} dx = \sqrt{\frac{\pi \gamma}{2}} e^{-i\pi/4}.
\]

(A.9)

A similar substitution of the power series (Eq. (A.7)) and the appropriate initial conditions into the differential equation for \( C_{+}(\tau) \) results (to second order in \( \sqrt{\gamma} \)) in

\[
\tilde{\ell} = 1 - \frac{\pi \gamma}{4}.
\]

(A.10)
Equations (A.9) and (A.10) coincide with Eq. (2.8) to second order in $\sqrt{\gamma}$, with $\theta_t$ and $\theta_r$ given by Eq. (2.9b). The procedure described above can be repeated for the initial conditions $C_-(\infty) = 1$ and $C_+(\infty) = 0$. Thus one verifies explicitly the other two elements of the scattering matrix $S$ (cf. Eq. (2.7)).

The Adiabatic Limit, $\gamma \gg 1$.

In this limit it is convenient to replace the representation of the wave function in Eq. (A.2) by the adiabatic basis, in which the Hamiltonian in Eq. (A.1) is instantaneously diagonalized,

$$\psi(t) = C_d(t) e^{i\phi_d(t)} |\psi_d(t)\rangle + C_u(t) e^{i\phi_u(t)} |\psi_u(t)\rangle; \quad (A.11)$$

here

$$H(t) |\psi_{ud}(t)\rangle = \varepsilon_{ud}(t) |\psi_{ud}(t)\rangle, \quad \varepsilon_{ud}(t) = \pm \frac{1}{2} \sqrt{\Delta^2 + (\Delta t)^2} \quad (A.12a)$$

$$\phi_{ud}(t) = \frac{\int_0^t \varepsilon_{ud}(t') dt'}{\hbar}. \quad (A.12c)$$

We impose the initial conditions $C_d(\infty) = 1$, $C_u(\infty) = 0$. Note that for $t \to -\infty$, $|\psi_d(\infty)\rangle = |+\rangle$, and $|\psi_u(\infty)\rangle = |-\rangle$, so that these initial conditions are identical to those imposed in the sudden limit. Also, at $t \to \infty$, we have $|\psi_d(\infty)\rangle = |-\rangle$ and $|\psi_u(\infty)\rangle = |+\rangle$, so that the definitions

$$\tilde{t} = C_u(\infty), \quad \tilde{\tau} = C_d(\infty) \quad (A.13)$$

are consistent with Eq. (A.3). In the adiabatic limit ($\gamma \gg 1$), the probability to undergo a transition from one adiabatic state to another is negligible (exponentially small). For the above initial conditions, this implies that one may approximate $C_d(t)$ by unity for all times. We thus have [46]

$$\tilde{t} = C_u(\infty) \approx \frac{\Delta \gamma}{8} \int_{-\infty}^{\infty} \frac{dt}{\varepsilon_u^2(t)} \exp \left\{ \frac{2i}{\hbar} \int_0^{\tilde{\tau}} \varepsilon_u(t') dt' \right\}$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \frac{dy}{y^2 + 1} \exp \left\{ i\gamma \int_0^{\infty} \sqrt{1 + x^2} \, dx \right\}; \quad (A.14)$$

in the last integral we rescale time so that $y \equiv t/(\Delta / \gamma)$. An explicit solution for integrals of this type was derived in previous works [47–48]. Below we sketch the main steps of the derivation, and refer to Refs. [47, 48] for further details. We define the variable

$$w \equiv \int_0^{\infty} dx \sqrt{x^2 + 1}. \quad (A.15)$$
In terms of $w$, the integrand on the right-hand side of Eq. (A.14) has simple poles in the complex plane at

$$w_c = \int_0^{y_e} dx \sqrt{x^2 + 1}, \quad \text{where} \quad y_c^2 + 1 = 0$$  \hspace{1cm} (A.16)

(i.e., at the zeros of $\varepsilon_e(t)$). To see this, we expand the integrand in Eq. (A.15) near $y_c$ to lowest order in $x - y_c$ and find

$$w \approx w_c + \frac{2\sqrt{2}}{3} (y - y_c)^{3/2}.$$  \hspace{1cm} (A.17)

Substituting in Eq. (A.14), we obtain

$$\tilde{r} \approx \frac{1}{2} \int_{-\infty}^{\infty} \frac{dw e^{i\gamma w}}{(dw/dy)^3},$$

where

$$\left(\frac{dw}{dy}\right)^3 \approx 3y_c(w - w_c) \quad \text{for} \quad w \to w_c.$$  \hspace{1cm} (A.18)

From Eq. (A.16), there are two points $y_c = \pm i$, which correspond to the poles $w_c = \pm i\pi/4$. We calculate the integral in Eq. (A.18) by closing a path around the upper half of the complex plane. This path encloses only the pole at $w_c = i\pi/4$. The integration then yields

$$\tilde{r} \approx \frac{\pi}{3} \exp\{i\gamma w_c\} = \frac{\pi}{3} \exp\left\{-\frac{\pi y}{4}\right\}.$$  \hspace{1cm} (A.19)

We have thus verified that in the adiabatic limit ($\gamma \gg 1$) the transmission coefficient $\tilde{r}$ is exponentially small. Consequently, we have confirmed the assumption that $C_d(t)$ is unity up to exponentially small correction, which implies $\tilde{r} \approx 1$ (cf. Eq. (A.13)). Equation (A.19) is equivalent to Eq. (2.8) up to the prefactor $\pi/3$, with $\theta_1$ and $\theta_r$ given by Eq. (2.9a).

**APPENDIX B: EXTENSION OF THE LANDAU–ZENER PROBLEM TO A DOUBLE GAP SPECTRUM**

We consider the Hamiltonian $H_0(t)$ in Eq. (2.1), and calculate the asymptotic transition probability $P$ defined in Eq. (2.4). We first study the sudden limit, $\delta \ll 1$, 
where \( \delta \) is defined in Eq. (2.13). Similarly to what has been done in Appendix A, we insert the wave function \( \psi(t) \) from Eq. (2.3) into the time-dependent Schrödinger equation and obtain two coupled equations for the coefficients \( C_{\pm}(t) \):

\[
\dot{C}_{\pm}(t) = \frac{iA}{2h} \exp \left\{ \pm \frac{i(\alpha t^3/3 - \varepsilon t)}{\hbar} \right\} C_{\pm}(t). \tag{B.1}
\]

In terms of the dimensionless time \( y \equiv (x/h)^{1/3} t \), Eq. (B.1) becomes

\[
\frac{dC_{\pm}(y)}{dy} = \frac{i\delta}{2} e^{\pm i(\varepsilon^{1/3} - \tilde{\varepsilon} y)} C_{\pm}(y), \quad \tilde{\varepsilon} \equiv \frac{\varepsilon}{(h^2 x)^{1/3}}. \tag{B.2}
\]

We expand the solutions \( C_{\pm}(t) \) in powers of the small parameter \( \delta \) and impose the initial conditions \( C_{-}(-\infty) = 1, C_{+}(-\infty) = 0 \). After integrating over \( y \) we obtain, to first order in \( \delta \),

\[
C_{+}(\infty) \approx \frac{i\delta}{2} \int_{-\infty}^{\infty} e^{i(\varepsilon^{1/3} - \tilde{\varepsilon} y)} = 2\pi Ai\{ -\tilde{\varepsilon} \}, \tag{B.3}
\]

where, in the last equality, we use the integral representation of the Airy function [49]. This implies that the transition probability in the sudden limit is given by Eq. (2.14b). In the limit \( \tilde{\varepsilon} \gg 1 \) (which, as explained in Section 2, implies \( \tau_z \ll t_e \)), we use the asymptotic form of the Airy function [49] and obtain

\[
P \approx \frac{\pi A^2}{2h \sqrt{\pi \varepsilon}} [1 - \cos(\pi/2 + 4\tilde{\varepsilon}^{3/2}/3)]. \tag{B.4}
\]

Expanding \( P_{LZ} \) in Eq. (2.6) to first order in the exponent and inserting Eqs. (2.6), (2.9b), and (2.11) into Eq. (2.12), we verify that \( P \) in Eq. (2.14b) coincides with the "separable" expression (Eq. (2.12) for \( \tau_z \ll t_e \)).

We next consider the adiabatic limit, \( \delta \gg 1 \). We follow exactly the same procedure introduced in Appendix A for the adiabatic approximation and represent the wavefunction in the form of Eq. (A.11). Here the adiabatic states are eigenstates of \( H_0(t) \) (Eq. (2.1)), and the corresponding adiabatic eigenvalues are given by Eq. (2.2). In this case the adiabatic states \( |\psi_{ud}(t)\rangle \) satisfy \( |\psi_{ud}(\pm \infty)\rangle = |+\rangle \) and \( |\psi_{ud}(\pm \infty)\rangle = |\pm\rangle \), where \( |\pm\rangle \) are the diabatic states (employed above to represent the wave function in the sudden limit). For the initial conditions \( C_{ud}(-\infty) = 1 \) and \( C_{ud}(-\infty) = 0 \), the asymptotic transition probability is

\[
P = |C_{ud}(\infty)|^2. \tag{B.5}
\]
Similarly to what has been done in Appendix A, we employ the adiabatic approximation in which

\[ C_u(\infty) \approx \int_{-\infty}^{\infty} dy \frac{y}{(1 + (s - y^2)^2)^{3/2}} \exp \left\{ i \delta^{3/2} \int_0^\infty dx \sqrt{1 + (s - x^2)^2} \right\}; \quad (B.6) \]

here \( y \equiv \sqrt{\alpha/\Delta(t)} \), \( s \equiv \varepsilon/\Delta \).

Note that the integral in Eq. (B.6) has a structure similar to the expression in Eq. (A.14), with the polynomial \( 1 + x^2 \) replaced by \( 1 + (s - x^2)^2 \). Similarly to the derivation following Eq. (A.14), we then employ the method of Refs. [47, 48] and define the variable \( w \), where here

\[ w \equiv \int_0^y dx \sqrt{1 + (s - x^2)^2}. \quad (B.7) \]

The corresponding critical points in the complex \( w \)-plane are

\[ w_{(\pm, \pm)} = \int_0^{y_{(\pm, \pm)}} dx \sqrt{1 + (s - x^2)^2}, \quad y_{(\pm, \pm)} = \pm (s \pm i)^{1/2}. \quad (B.8) \]

In the vicinity of a critical point we find

\[ w - w_{(\pm, \pm)} = \frac{2 \sqrt{4(y_{(\pm, \pm)}^2 - s)} y_{(\pm, \pm)} (y - y_{(\pm, \pm)})^{3/2}}{3}, \quad (B.9) \]

and, consequently,

\[ C_u(\infty) \approx \int_{-\infty}^{\infty} \frac{dwe^{i\delta^{3/2}w}y(w)}{(dw/dy)^3}, \]

where

\[ \left( \frac{dw}{dy} \right)^3 \approx 6y_{(\pm, \pm)}(y_{(\pm, \pm)}^2 - s)(w - w_{(\pm, \pm)}) \quad \text{for} \quad w \to w_{(\pm, \pm)}. \quad (B.10) \]

Similarly to the integration performed in Appendix A, we close a path in the upper half plane, which, in this case, encloses two simple poles: \( w_{(+, +)} \) and \( w_{(-, -)} \). These poles are related by

\[ \Re(w_{(+, +)}) = -\Re(w_{(-, -)}), \quad \Im(w_{(+, +)}) = \Im(w_{(-, -)}). \quad (B.11) \]

Here \( \Re(z) (\Im(z)) \) denote the real (imaginary) part of \( z \). We hence obtain

\[ P = |C_u(\infty)|^2 \approx \left( \frac{\pi}{3} \right)^2 e^{2R(\delta, s)} 4 \sin^2(I(\delta, s)), \quad (B.12a) \]
where

\[ R(\delta, s) \equiv \Re (i \delta^{3/2} w_{(\pm, +)}(s) - i \delta^{3/2} \int_0^{(s+i)^{1/2}} \sqrt{1 + (s - y^2)^2} \, dy) \]  \hspace{1cm} (B.12b)

and

\[ I(\delta, s) \equiv \Im (i \delta^{3/2} w_{(\pm, +)}(s) - i \delta^{3/2} \int_0^{(t+i)^{1/2}} \sqrt{1 + (s - y^2)^2} \, dy) \]  \hspace{1cm} (B.12c)

The definite integrals in Eqs. (B.12b) and (B.12c) were calculated by Crothers and Hughes [32], employing the substitution \( y \equiv (s + i)^{1/2} \sin \theta \):

\[
\int_0^{(s+i)^{1/2}} \sqrt{1 + (s - y^2)^2} \, dy = e^{i\pi/4} (1 - is)(1 + is)^{1/2} \int_0^{\pi/2} d\theta \cos^2 \theta \left( 1 + \frac{(1-is) \sin^2 \theta}{(1+is)} \right)^{1/2}. \hspace{1cm} (B.13)
\]

Applying the integral representation of the hypergeometric function (Eq. (15.3.1) in Ref. [49], with the substitution \( \sin^2 \theta \equiv t \)), and the identities (8.1.4) and (15.4.14) of Ref. [49], we obtain Eq. (2.14a).

The expressions for \( R(\delta, s) \) and \( I(\delta, s) \) are considerably simplified in the limit \( s \gg 1 \). In that limit we employ the asymptotic form of \( {}_2F_1(a, b; c; z) \) for large \( z \) (Eq. (15.3.14) in Ref. [49]). Using the identities (6.3.7) and (6.1.18) of Ref. [49] for the functions \( I(z) \) and \( \psi(z) \), we obtain (for \( s \gg 1 \))

\[ R(\delta, s) \approx -\frac{\pi \delta^{3/2}}{8s^{1/2}} = -\frac{\pi A^2}{4h\varepsilon_{\text{eff}}}, \hspace{1cm} (B.14a) \]

\[ I(\delta, s) \approx \frac{2(s\delta)^{3/2}}{3} = \frac{2\varepsilon^{3/2}}{3h\varepsilon^{1/2}}. \hspace{1cm} (B.14b) \]

Inserting Eq. (B.14) into Eq. (B.12b), we obtain the separable expression in Eq. (2.12). As explained in Section 2, the condition \( s \gg 1 \) is equivalent to \( \tau_z \ll t_z \), where \( \tau_z \) is calculated in the adiabatic limit. We conclude that Eq. (2.12) is a good approximation for \( P \), for \( \tau_z \ll t_z \), in the adiabatic as well as in the sudden limits.

**APPENDIX C: DEPHASING DUE TO NON-DYNAMICAL ENVIRONMENT**

Here we couple our toy model (the Hamiltonian of Eq. (2.1)) to an environment consisting of two-level degrees of freedom. The system and environment are described by Eqs. (3.1) and (3.3), and the corresponding wave function, \( \Psi(t) \), is depicted in Eq. (3.4). Below we derive the asymptotic transition probability for this
system. Imposing $C_-(-\infty) = 1, C_+(-\infty) = 0$ as initial conditions for our system, the transition probability $P_{\text{eff}}$ is defined as
\begin{equation}
P_{\text{eff}} = |C_+(\infty)|^2.
\end{equation}
We demonstrate the phase destruction mechanism due to the coupling to the environment and obtain an explicit expression for the dephasing time, $\tau_\phi$, in a simple case.

We insert Eqs. (3.4), (3.1), and (3.3) into the time-dependent Schrödinger equation
\begin{equation}
i\hbar \frac{\partial \Psi(t)}{\partial t} = H(t) \Psi(t).
\end{equation}
We define a set of variables $U^K_{\pm}(t)$ as
\begin{equation}
U^K_{\pm}(t) = C_{\pm} \prod_{k \in K} a_k(t) \prod_{k \notin K} b_k(t) \exp \left\{ \frac{\mp i}{2\hbar} \left[ \sum_{k \in K} (V_k \pm \zeta_k) t - \sum_{k \notin K} (V_k \pm \zeta_k) t \right] \right\},
\end{equation}
where $K$ is a certain subset of $\{1, ..., N\}$. Using the definitions of the Pauli matrices,
\begin{equation}
\sigma_z |\pm\rangle = \pm |\pm\rangle, \quad \sigma_x |\pm\rangle = |\mp\rangle, \quad \sigma_z^{(k)} |\pm\rangle^{(k)} = \pm |\pm\rangle^{(k)},
\end{equation}
and equating coefficients of the $2^{N+1}$ states
\begin{equation}
|\pm\rangle \prod_{k=1}^{N} |\pm\rangle^{(k)},
\end{equation}
we obtain a set of coupled equations for $U^K_{\pm}(t)$:
\begin{equation}
i\hbar U^K_{\pm}(t) = \frac{\Delta}{2} U^K_{\mp}(t) \exp \left\{ \frac{\mp i}{\hbar} \left[ \left( \varepsilon - \sum_{k \in K} V_k + \sum_{k \notin K} V_k \right) t - \frac{\alpha t^3}{3} \right] \right\}.
\end{equation}
Note that Eq. (C.6) couples each pair of variables $U^K_{\pm}(t)$ of the same $K$ only. This equation is identical to Eq. (B.1), with $C^\pm_{\pm}(t)$ being replaced by $U^K_{\pm}(t)$, and the parameter $\varepsilon$ being replaced by $(\varepsilon - \sum_{k \in K} V_k + \sum_{k \notin K} V_k)$. Hence, the asymptotic solution $|U^K_+(\infty)|^2$, for the initial conditions $U_-^K(-\infty) = U^K_0$ and $U_+^K(-\infty) = 0$, is
\begin{equation}
|U^K_+(\infty)|^2 = |U^K_0|^2 P(\varepsilon'),
\end{equation}
\begin{equation}
\varepsilon' \equiv \varepsilon - \sum_{k \in K} V_k + \sum_{k \notin K} V_k,
\end{equation}
\begin{equation}
|U^K_0| \equiv \prod_{k \in K} |a_k(-\infty)| \prod_{k \notin K} |b_k(-\infty)|.
We have introduced here the notation \( P(\varepsilon) \), to stress the fact that \( P \) (see, e.g., Eqs. (2.14)) depends on \( \varepsilon \). The last equality in Eq. (C.7) follows from the definition of \( U^K_\pm(t) \); the transition probability \( P(\varepsilon) \) was obtained in Section 2 (with a detailed derivation in Appendix B). We then employ the normalization conditions
\[
|a_k(t)|^2 + |b_k(t)|^2 = 1, \tag{C.8}
\]
which implies (for all \( t \))
\[
|C_+(t)|^2 = \sum_K |U^K_+(t)|^2; \tag{C.9}
\]
here \( \sum_K \) denotes the sum over all possible choices of a subset of \( \{1, ..., N\} \). Inserting Eq. (C.7) into Eq. (C.9) (with \( t \to \infty \)) yields Eq. (3.5).

An explicit expression for \( P_{\text{eff}} \) can be derived analytically employing certain simplifying assumptions. We consider Eq. (3.5) with \( V_k = V \), \( a_k(-\infty) = a_0 \), and \( b_k(-\infty) = b_0 \) for all \( k \). Then, all subsets \( K \) of the same number of degrees of freedom, \( n \), have an identical contribution to \( \sum_K \). Equation (3.5) in this case reduces to
\[
P_{\text{eff}} = \sum_{n=0}^N \binom{N}{n} p^n (1 - p)^{N-n} P(\varepsilon + V(N-2n)). \tag{C.10}
\]
For large \( N \) and \( 1 \ll pN, 1 \ll (1 - p)N \), and defining
\[
p \equiv \min\{|a_0|^2, |b_0|^2\} \tag{C.11}
\]
with \( |a_0|^2 + |b_0|^2 = 1 \), one can approximate the binomial factors in Eq. (C.10) by a Gaussian distribution:
\[
\binom{N}{n} p^n (1 - p)^{N-n} \approx \frac{1}{\sqrt{2\pi npN}} \exp \left\{ - \frac{(n - pN)^2}{2pN} \right\}. \tag{C.12}
\]
We further assume that the coupling is weak (\( VN \ll \varepsilon \)) and that the condition of "separability" (\( \tau_c \ll t_s \)) is satisfied. As explained in Section 3, under these assumptions we can approximate \( P(\varepsilon + V(N-2n)) \) in Eq. (C.10) by Eq. (2.12), and to lowest order in \( NV/\varepsilon \) keep the \( V \)-dependence of \( P(\varepsilon + V(N-2n)) \) (to first order) in the dynamical phase only. We thus obtain
\[
P_{\text{eff}} \approx 2P_{\text{LZ}}(1 - P_{\text{LZ}}) \left[ 1 - \sum_v D(\nu) \cos \left\{ \frac{(4\varepsilon^{3/2}/3 + 2\nu^{1/2})}{\hbar \alpha^{1/2}} - 2\theta \right\} \right]; \tag{C.13}
\]
here
\[
v \equiv 2V(n - pN)
\]
and

\[ D(v) = \frac{1}{\sqrt{2\pi v_0}} \exp \left\{ -\frac{v^2}{2v_0^2} \right\}, \quad v_0 = 2V \sqrt{pN}. \]

Employing the symmetry of \( D(v) \) with respect to \( v \to -v \), we obtain

\[ P_{\text{eff}} \approx 2P_{LZ} (1 - P_{LZ}) \left[ 1 - \cos \left\{ \frac{4e^{3/2}}{3h^2 \alpha^{1/2}} - 2\theta_r \right\} \sum_v D(v) \cos \left\{ \frac{t_z v}{h} \right\} \right]. \quad (C.14) \]

For large \( N \), it is justified to change the summation over \( v \) in Eq. (C.14) into an integral. The result of this integration (employing the definition of \( p \) (Eq. (C.11))), is Eq. (3.9), with the dephasing time \( \tau_\phi \) given by Eq. (3.10).

**APPENDIX D: DEPHASING DUE TO NON-DYNAHMICAL ENVIRONMENT: A POWER LAW DISTRIBUTION OF COUPLING STRENGTHS**

We consider here the transition probability \( P_{\text{eff}} \), in the presence of a non-dynamical environment, with a power law distribution of coupling coefficients. In general, we have shown in Section 3 that \( P_{\text{eff}} \) is given by Eq. (3.5). Assuming that \( a_k(-\infty) = b_k(-\infty) = 1/\sqrt{2} \) for all \( k \), then the expression in Eq. (3.5) can be rewritten in the form of Eq. (3.6), with

\[ D(v) = \delta \left( v - \sum_{k=1}^{N} V_k \right); \quad (D.1) \]

here \( \delta(x) \) is the Dirac \( \delta \)-function, and \( \langle \rangle \) denotes averaging over the distribution \( p(V) \) of the random variables \( V_k \). Equations (3.6) and (D.1) are compatible with Eq. (3.5), provided that \( p(V) = p(-V) \). Using the integral representation of the \( \delta \)-function in Eq. (D.1), we obtain

\[ D(v) = \int_{-\infty}^{\infty} \prod_{k=1}^{N} dV_k \prod_{k=1}^{N} p(V_k) \int_{-\infty}^{\infty} \frac{dy}{2\pi} \exp \left\{ iv \left( v - \sum_{k=1}^{N} V_k \right) \right\} \]

\[ = \int_{-\infty}^{\infty} \frac{dy}{2\pi} \exp \left\{ i\bar{v}v + N \ln \left( \int_{-\infty}^{\infty} dV p(V) e^{-bV} \right) \right\}. \quad (D.2) \]

We now consider the limit \( \tau_z \ll t_z \), where consecutive transition are decoupled. In addition, we employ the assumption of weak coupling. The probability \( P_{\text{eff}} \) is then approximated by Eq. (3.7). The symmetry \( p(V) = p(-V) \) implies \( D(v) = D(-v) \) (cf. Eq. (D.2)), hence we obtain (for a general symmetric distribution)

\[ P_{\text{eff}} \approx 2P_{LZ} (1 - P_{LZ}) \left[ 1 - F_A \cos \left\{ \frac{4e^{3/2}}{3h^2 \alpha^{1/2}} - 2\theta_r \right\} \right], \]

where \( F_A \) is given by Eq. (3.3).
where
\[ F_A = \int_{-\infty}^{\infty} dv \, D(v) \cos \left( \frac{t_s v}{\hbar} \right). \]  (D.3)

Equations (D.2) and (D.3) imply
\[ F_A = \left( \int_{-\infty}^{\infty} dV p(V) e^{-i V/V_c} \right)^N. \]  (D.4)

Employing Eq. (D.2) for the particular case where \( p(V) \) is given by Eq. (3.12), we find
\[ D(v) \sim \int_{-\infty}^{\infty} \frac{dy}{2\pi} \, e^{ivy} \left| \frac{1}{y} \right|^{N(\gamma + 1)} \sim |v|^{-1 + N(\gamma + 1)}. \]  (D.5)

(if the integral exists). For the attenuation factor we obtain (inserting Eq. (3.12) into Eq. (D.4))
\[ F_A = \left( \int_{-\infty}^{\infty} dV |V|^\gamma e^{-i V/V_c} \right)^N \sim \left| \frac{1}{V_c} \right|^{N(\gamma + 1)}. \]  (D.6)

We have thus verified Eq. (3.13).

To guarantee normalizability of the distribution \( p(V) \), we next introduce cutoffs. We impose an upper cutoff \( V_c \) for \( \gamma \geq -1 \), and a lower cutoff \( V_0 \) for \( \gamma \leq -1 \). We consider first the case \( \gamma \geq -1 \) and assume
\[ p(V) = C |V|^\gamma e^{-V/V_c}, \]  (D.7)
\[ C = \frac{2^\gamma \Gamma(\gamma/2 + 1)}{\sqrt{\pi} \Gamma(\gamma + 1)} V_c^{\gamma + 1}; \]
\[ \int dV p(V) = 1. \]  

Equation (D.4) implies that the attenuation factor \( F_A \), for a symmetric distribution, is determined by the Fourier transform of \( p(V) \). Substituting Eq. (D.7) in (D.4), we obtain
\[ F_A = \left( \frac{(\sqrt{2})^{\gamma - 1} \Gamma(\gamma/2 + 1)}{\sqrt{\pi}} \exp \left\{ - \frac{(V_c t_s)^2}{8h^2} \right\} 2\Re \left\{ D_{-\gamma - 1} \left( \frac{i V_c t_s}{\sqrt{2h}} \right) \right\} \right)^N; \]  (D.8)

here \( D_{-\gamma - 1}(z) \) is the parabolic cylinder function [50]. We then distinguish between the behavior of \( F_A \) for short and long time scales \( t_s \), defined with respect to a
characteristic time, $h/V_c$. For $t_s \ll h/V_c$, we use the small argument expansion of $D_{-γ-1}(z)$ [49, 50], and obtain

$$F_A \approx \left[1 - \left(\frac{γ + 1/2}{4}\right)\left(\frac{V_c t_s}{h}\right)^2\right]^N \exp \left\{ -N\left(\frac{V_c t_s}{h}\right)^2 \right\}$$

$$\approx \exp \left\{ -\frac{t_s^2}{τ_φ^2} \right\}, \quad τ_φ \equiv \frac{2h}{V_c \sqrt{N(γ + 1)}}. \quad \text{(D.9)}$$

For $t_s \gg h/V_c$, we use the asymptotic form of the parabolic cylinder function for large arguments [49, 50] and find

$$F_A \approx \left| \frac{τ_φ}{t_s} \right|^{N(γ + 1)}, \quad \text{(D.10)}$$

where

$$τ_φ \equiv \frac{2h}{V_c} \left(\frac{Γ(γ/2 + 1) \cos[π(γ + 1)/2]}{\sqrt{π}}\right)^{1/(γ + 1)}.$$

We thus conclude that for short time-scales, one recovers the exponential form of $F_A$ in Eq. (3.9), whereas for long time-scales we obtain the power-law decay (cf. Eq. (3.13)). Note that in the limit $V_c \to \infty$, we recover Eq. (3.13) for essentially all values of $t_s$.

We next consider the case $γ \leq -1$, and introduce a lower cutoff $V_0$ in the distribution $p(V)$. For convenience, we choose the form

$$p(V) = C(V^2 + V_0^2)^{γ/2}, \quad \text{(D.11)}$$

with

$$C = \frac{2^γ Γ(γ - 1) \sin[-π(γ + 1)/2]}{\sqrt{π/2}V_0^{γ + 1}}.$$

Inserting Eq. (D.11) into (D.4), we obtain

$$F_A = \left[\frac{2^{γ+2} Γ(γ - 1) \sin[-π(γ + 1)/2]}{Γ(γ/2) \sqrt{πV_0^{γ + 1}}}(\frac{t_s}{2hV_0})^{-(γ + 1)/2} K_{-(γ + 1)/2}\left(\frac{V_0 t_s}{h}\right)\right]^N; \quad \text{(D.12)}$$

where $K_{-(γ + 1)/2}(z)$ is the modified Bessel function [50]. In the limit $t_s \ll h/V_0$, we use the asymptotic form of $K_{-(γ + 1)/2}(z)$ for small $z$, and obtain for $-3 < γ < -1$

$$F_A \approx 1 - N\left(\frac{τ_φ}{t_s}\right)^{γ + 1}, \quad τ_φ \equiv \frac{2h}{V_0}. \quad \text{(D.13)}$$
For \( t_s \gg h/V_0 \), we employ the asymptotic form [49] of \( K_{-(\gamma+1)/2}(z) \) for large \( z \). We obtain

\[
F_A \approx \left[ \frac{2\gamma^2 e^{-\gamma^2} \sin[-\pi(\gamma+1)/2]}{\Gamma(-\gamma/2)} \left( \frac{V_0 t_s}{h} \right)^{-(\gamma^2+1)/2} \right]^N \exp \left\{ -\frac{t_s}{\tau_\phi} \right\}, \quad (D.14)
\]

where \( \tau_\phi \equiv h/NV_0 \). From Eqs. (D.13) and (D.14) we find that for short time-scales, \( F_A \) is dominated by a power-law in \( t_s \), whereas for long time-scales it is dominated by an exponential dependence on \( t_s \). However, unlike the results obtained for \( \gamma \geq -1 \) we do recover neither Eq. (3.9) nor Eq. (3.13) in any of the regimes. We note that the expression for \( F_A \) in Eq. (3.9) was underlined by the central limit theorem; this is not the case for \( p(V) \) of Eq. (D.11).

**APPENDIX E: DEPHASING DUE TO A HARMONIC BATH**

We consider the dynamics of the system described by \( H_0(t) \) (Eq. (2.1)), coupled to a bath which consists of harmonic degrees of freedom. The coupling to the bath is described by Eqs. (3.1) and (3.14), and the bath is prepared in a thermal state at temperature \( T \). Below we calculate the asymptotic transition probability, \( P_{\text{eff}} \), and demonstrate the dephasing effects induced by this coupling. Our model is identical to the spin-boson model considered in Ref. [22], except that in our case the bias (the coefficient of \( \sigma_z \) in \( H_0(t) \)) is time-dependent [51], \( \alpha t^2 - \varepsilon \). Note also that our definition of \( P_{\text{eff}} \) coincides with \( (1 - P(\infty))/2 \), where \( P(t) \) is defined and evaluated (for a constant bias) in Ref. [22]. We therefore repeat the derivation of Ref. [22] (see for details) and obtain

\[
P_{\text{eff}} = \sum_{n=1}^{\infty} (-1)^{n+1} \Delta^{2n} K_n,
\]

where

\[
K_n \equiv 2^{-n-1} \sum_{\{\zeta_i\}} \int_{-\infty}^{\infty} dt_{2n} \int_{-\infty}^{t_{2n}} dt_{2n-1} \cdots \int_{-\infty}^{t_2} dt_1 F_n(t_1, \ldots, t_{2n}; \zeta_1, \ldots, \zeta_n); \quad (E.1)
\]

here

\[
F_n \equiv F_1 F_2 F_3 F_4, \quad (E.2)
\]

with

\[
F_1 \equiv \exp \left\{ -\frac{4}{\pi h} \sum_{j=1}^{n} S_j \right\}, \quad (E.3a)
\]

\[
F_2 \equiv \exp \left\{ -\frac{4}{\pi h} \sum_{k=1}^{n} \sum_{j=k+1}^{n} \zeta_j \zeta_k A_{jk} \right\}, \quad (E.3b)
\]
\[ F_3 = \prod_{k=1}^{n} \cos \left\{ \frac{4}{\pi \hbar} \sum_{j=k+1}^{n} \zeta_j X_{jk} \right\}. \]  
(E.3c)

\[ F_4 \equiv \cos \left\{ \sum_{j=1}^{n} \zeta_j \left[ \frac{\alpha}{3\hbar} (t_j^2 - t_{2j-1}^2) - \frac{\varepsilon}{\hbar} (t_{2j} - t_{2j-1}) - \frac{4}{\pi \hbar} X_{j0} \right] \right\}. \]  
(E.3d)

The quantities \( S_j, A_{jk}, \) and \( X_{jk} \) are functions of \( \{t_m\} \) and depend on the spectral density \( J(\omega) \):

\[
S_j \equiv Q_2(t_{2j} - t_{2j-1}),
\]

\[
A_{jk} \equiv Q_2(t_{2j} - t_{2k-1}) + Q_2(t_{2j-1} - t_{2k}) - Q_2(t_{2j-1} - t_{2k-1}),
\]

\[
X_{jk} \equiv Q_1(t_{2j} - t_{2k+1}) + Q_1(t_{2j-1} - t_{2k}) - Q_1(t_{2j-1} - t_{2k+1}),
\]

where

\[
Q_1(t) \equiv \int_{0}^{\infty} \frac{J(\omega)}{\omega^2} \sin(\omega t) \, d\omega,
\]

\[
Q_2(t) \equiv \int_{0}^{\infty} \frac{J(\omega)}{\omega^2} (1 - \cos(\omega t)) \coth(\beta \hbar \omega/2) \, d\omega, \quad \beta = \frac{1}{k_B T}.
\]

(E.5a)

(E.5b)

Here \( \zeta_j = \pm 1 \) are the signs of the "blips," which represent off-diagonal states of the density matrix of the system [22, 40]. The above result differs from those of Refs. [22, 40] by the expression for \( F_4 \), which includes the dynamical phase contributed by the bias (the first two terms in the square brackets of Eq. (E.3d)).

We then consider \( H_0(t) \) in the sudden limit, \( \delta \ll 1 \) (cf. Eq. (2.13)). In this limit we keep only second order in \( \Delta \) in Eq. (E.1), hence the expression for \( P_{\text{eff}} \) is considerably simplified. In particular, the contribution of interactions between different "blips" [22] vanish; the reason is that in this limit we include only single-blip paths. The approximate expression for \( P_{\text{eff}} \) is

\[
P_{\text{eff}} \approx \frac{\Delta^2}{2\hbar^2} \int_{-\infty}^{\infty} dt_2 \int_{-\infty}^{t_2} dt_1 \cos \left\{ \frac{\alpha}{3\hbar} (t_2^3 - t_1^3) - \frac{\varepsilon}{\hbar} (t_2 - t_1) \right\} \]

\[
\times \cos \left\{ \frac{4}{\pi \hbar} Q_1(t_2 - t_1) \right\} \exp \left\{ - \frac{4}{\pi \hbar} Q_2(t_2 - t_1) \right\}.
\]

(E.6)

We employ the change of variables

\[
T \equiv \frac{t_1 + t_2}{2}, \quad \tau \equiv (t_2 - t_1);
\]

(E.7)
Eq. (E.6) then becomes

\[
P_{\text{eff}} \approx \frac{A^2}{4h^2} \int_{-\infty}^{\infty} d\tau \cos \left\{ \frac{4}{\pi h} Q_1(\tau) \right\} \exp \left\{ -\frac{4}{\pi h} Q_2(\tau) \right\} \\
\times \int_{-\infty}^{\infty} dT \cos \left\{ \frac{T}{\hbar} \left[ \varepsilon - \alpha \left( T^2 + \frac{T^2}{12} \right) \right] \right\}.
\] (E.8)

We integrate over \( T \) first (this is a Gaussian integral), and obtain Eq. (3.16a). Assuming an ohmic spectrum, we use Eq. (3.15) for the spectral density \( J(\omega) \) and insert in Eq. (E.5). We thus obtain the explicit expressions for \( Q_1(t) \), \( Q_2(t) \) (Eqs. (3.16b) and (3.16c)).

To obtain an explicit expression for \( P_{\text{eff}} \), we consider Eq. (3.16) in the limit \( \tau_s \ll \tau_s \). This corresponds in the sudden limit to \( \varepsilon \gg (\hbar^2 \alpha)^{1/3} \), which implies that we can evaluate the integral in Eq. (3.16a) in a stationary phase approximation. We rewrite the integral in terms of the new variable \( x \equiv \tau^{1/2} \):

\[
P_{\text{eff}} \approx \frac{A^2 \pi^{1/2}}{16h^{3/2} \alpha^{1/2}} \left[ \int_{C_1} (e^{g_+^{\ast}(x)} + e^{g_-^{\ast}(x)}) \, dx + \int_{C_2} (e^{g_+^{\ast}(x)} + e^{g_-^{\ast}(x)}) \, dx \right], \quad (E.9a)
\]

\[
g_\pm(x) \equiv i \left\{ \frac{\varepsilon x^2}{\hbar} \mp \frac{\alpha x^6}{12\hbar} + \frac{\pi}{4} \pm 2\gamma \arctan(\omega_c x^2) \right\} \\
- 2\gamma \ln \left\{ \sqrt{1 + \omega_c^2 x^4} \frac{\sinh(\pi x^2/\beta \hbar)}{(\pi x^2/\beta \hbar)} \right\}; \quad (E.9b)
\]

here \( \gamma \equiv 2\eta/\pi \hbar \), and \( C_1, C_2 \) are drawn in Fig. 8a. We next find the saddle points, which are the roots of the equations

\[
g^{'}_\pm(x) = i \left\{ \frac{\varepsilon}{\hbar} - \frac{\alpha x^4}{4\hbar} \pm \frac{2\gamma \omega_c}{1 + \omega_c^2 x^4} \right\} 2x \\
- 2\gamma \left\{ \frac{\omega_c^2 x^2}{1 + \omega_c^2 x^4} + \frac{\pi}{\beta \hbar} \left( \coth \left( \frac{\pi x^2}{\beta \hbar} \right) - \frac{\beta \hbar}{\pi x^2} \right) \right\} 2x = 0 \quad (E.10)
\]

(the prime denotes differentiation with respect to \( x \)). One of the saddle points is

\[
x_0 = 0, \quad (E.11)
\]

for which

\[
g_\pm(x_0) = -\frac{i\pi}{4} \quad (E.12)
\]

and

\[
g^{''}_\pm(x_0) = \frac{2i}{\hbar} (\varepsilon \pm 2\gamma \hbar \omega_c). \quad (E.13)
\]
Fig. 8. The paths $C_1, C_2$ (Fig. 8a) and $C_a, C_b$ (Fig. 8b) in the complex $x$-plane (cf. Appendix E). The points $x_0$ and $x_{\pm}^{(n)}$, with $n = 1, 2, 3, 4$, are the saddle points of the functions $g_{\pm}(x)$ and $g_{\pm}^*(x)$. 
To find the other saddle points, we consider the weak coupling limit, defined by Eq. (3.18). In this limit we write these saddle points as

\[ x_{\pm}^{(n)} = x_n + \delta x_{\pm}^{(n)}, \]  
(E.14a)

where

\[ x_{1(3)} = +(-) \sqrt{\frac{2}{\alpha}} \left( \frac{\gamma}{\alpha} \right)^{1/4}, \quad x_{2(4)} = +(-) i \sqrt{\frac{2}{\alpha}} \left( \frac{\gamma}{\alpha} \right)^{1/4}; \]  
(E.14b)

\( x_n \) in Eq. (E.14b) are the roots of Eq. (E.10) with \( \gamma = 0 \). We insert Eq. (E.14a) into Eq. (E.10) and find a solution to \( \delta x_{\pm}^{(n)} \) to linear order in the small parameters \( \gamma/\beta \epsilon, \gamma h \omega_c/\epsilon \) (cf. Eq. (3.18)). We obtain

\[ \delta x_{\pm}^{(n)} = \frac{\hbar}{\alpha x_n^3} \left\{ \frac{1}{\pm} \frac{2\gamma \omega_c}{1 + 4\omega_c^2 \epsilon/\alpha} \right\} \]
\[ - \frac{2\gamma h}{i \alpha x_n^3} \left\{ \frac{\omega_c^2 x_n^2}{1 + 4\omega_c^2 \epsilon/\alpha} + \frac{\pi}{\beta h} \left( \frac{\coth \left( \frac{\pi x_n^2}{\beta h} \right) - \frac{\beta h}{\pi x_n^2}}{\beta h} \right) \right\}. \]  
(E.15)

To further simplify the above expression, we impose additional constraints on the parameters, namely Eqs. (3.17); note that \( t_s = |x_n|^2 \). We then find

\[ x_{\pm}^{(n)} \approx x_n \left( 1 + \frac{i \pi \gamma}{2 \beta \epsilon} \right). \]  
(E.16)

The corresponding values of \( g_{\pm}(x) \) and their second derivatives, to first order in \( \gamma/\beta \epsilon, \gamma h \omega_c/\epsilon \), are

\[ g_{\pm}(x_{\pm}^{(n)}) \approx (-1)^{-n-1} \left\{ \frac{i 4 \epsilon^{3/2}}{3h x^{1/2} \pm i \pi} \right\} - \frac{4 \pi \gamma \epsilon^{1/2}}{\beta h x^{1/2} \pm i \pi} - \frac{4 \pi \gamma \epsilon^{1/2}}{4 - 2 \gamma \ln \left( \frac{\beta h \omega_c}{\pi} \right)}, \]  
(E.17)

\[ g_{\pm}''(x_{\pm}^{(n)}) \approx -i 8 \epsilon/\hbar. \]  
(E.18)

From Eq. (E.13) we find that the directions of steepest descent, in the vicinity of \( x_0 \), are at angles close to \( \pi/4, -3\pi/4 \) with respect to the positive real axis; similarly, from Eq. (E.18), the steepest descent directions near \( x_{\pm}^{(n)} \) are at angles close to \( -\pi/4, 3\pi/4 \) (cf. Fig. 8b). The dashed lines in Fig. 8b represent the asymptotes of the directions of stationary phase in the complex plane, over which the real part of \( g_{\pm}(x) \) is negative for \( |x| \to \infty \). These asymptotes are obtained from the limit \( |x| \to \infty \) of Eq. (E.9b) (in this limit, \( g_{\mp}(x) \) is dominated by the \( x^6 \) term). Similar considerations can be applied to the functions \( g_{\pm}(x) \). We then evaluate the integrals in Eq. (E.9a) by deforming the paths \( C_1, C_2 \) into \( C_a, C_b \), respectively (cf. Figs. 8a and b); the closed paths \( C_1 - C_a (C_2 - C_b) \) enclose regions in the complex
plane in which \( g_\pm (x) (g_\pm^*(-x)) \) are analytic. We note also that along the segments \( AA' \) and \( BB' \) (\( CC' \) and \( BB'' \)) (cf. Fig. 8b), \( g_\pm (x) (g_\pm^*(-x)) \) are large and negative. Consequently, we infer that the integrals in Eq. (E.9a) are dominated each by the contribution of three saddle points. The integral over \( C_1 \) is dominated by \( x_0 \) and \( x_+^{(1)-} \), \( x_+^{(1)} \) for the first (second) term, respectively; similarly, the integral over \( C_2 \) is dominated by \( x_0 \) and \( x_3^{+} \), \( x_3^{+} \). Each contribution is calculated according to a standard saddle point procedure, employing Eqs. (E.12), (E.13), (E.17), and (E.18). We thus obtain Eq. (3.19) (note that in the sudden limit \( \theta_r = -\pi/4 \), cf. Eq. (2.9b)). The effect of the bath is expressed by the attenuation factor \( F_A \), which multiplies the interference term of \( P_{\text{eff}} \).

APPENDIX F: THE EFFECT OF A COHERENT ENVIRONMENT ON DISSIPATION

We couple our toy model (Eq. (2.1)) to an environment of two-level degrees of freedom. The system and environment are described by the Hamiltonian of Eq. (4.1); the corresponding wave function, \( \psi(t) \), is represented in Eq. (3.4). This model for a coupling to external degrees of freedom differs from the model studied in Section 3(a) and Appendix C by the coupling term in the Hamiltonian. In particular, this term gives rise to dissipation, as discussed in Section 4. Below we derive the asymptotic transition probability \( P_{\text{eff}} \), defined in the same way as in Appendix C.

We insert Eqs. (3.4) and (4.1) into the time-dependent Schrödinger equation. Following the procedure described in Appendix C, we obtain a set of coupled equations for the variables \( U_\pm^K (t) \), defined by Eq. (C.3) with \( V_k = 0 \) for all \( k \):

\[
\frac{i\hbar}{\Delta} U_\pm^K (t) = \frac{\Delta}{2} U_\pm^K (t) \exp \left\{ \frac{\pm i}{\hbar} \left[ \frac{\alpha t^3}{3} \right] \right\} \\
- \frac{1}{2} \sum_{k \in K} V_k^{(x)} \left( \frac{b_k(t)}{a_k(t)} \right) U_\pm^K (t) \exp \left\{ \frac{\pm i}{\hbar} \left[ \frac{\alpha t^3}{3} \right] \right\} \\
- \frac{1}{2} \sum_{k \notin K} V_k^{(x)} \left( \frac{a_k(t)}{b_k(t)} \right) U_\pm^K (t) \exp \left\{ \frac{\pm i}{\hbar} \left[ \frac{\alpha t^3}{3} \right] \right\}.
\]  

Equation (F.1) couples many of the variables \( U_\pm^K (t) \), and some simplifying assumptions are required to enable an analytic solution. We therefore consider Eq. (4.1) in the sudden limit \( \delta \ll 1 \) (cf. Eq. (2.13)), and the weak coupling limit

\[
V_k^{(x)} \ll (\hbar^2 \alpha)^{1/3}.
\]  

We expand \( U_\pm^K (t) \) in powers of \( \delta \) and \( V_k^{(x)}/(\hbar^2 \alpha)^{1/3} \) and calculate \( U_\pm^K (\infty) \) for the initial conditions \( U_-^K (-\infty) = U_0^K \) and \( U_+^K (-\infty) = 0 \) (\( U_0^K \) is defined in Eq. (C.7)). After integrating Eq. (F.1) over \( t \), we obtain to first order in \( \delta \) and \( V_k^{(x)}/(\hbar^2 \alpha)^{1/3} \).
\[ i\hbar U_+^K(\infty) = \frac{A}{2} U_-^\kappa(-\infty) \int_{-\infty}^{\infty} dt \exp \left\{ \frac{i}{\hbar} \left[ \epsilon t - \frac{a t^3}{3} \right] \right\} \]

\[ -\frac{1}{2} \sum_{k \in K} V_k^{(x)} \left( \frac{b_k(-\infty)}{a_k(-\infty)} \right) U_-^\kappa(-\infty) \int_{-\infty}^{\infty} dt \exp \left\{ \frac{i}{\hbar} \left[ (\epsilon - \zeta_k) t - \frac{a t^3}{3} \right] \right\} \]

\[ -\frac{1}{2} \sum_{k \neq k} V_k^{(x)} \left( \frac{a_k(-\infty)}{b_k(-\infty)} \right) U_-^\kappa(-\infty) \int_{-\infty}^{\infty} dt \exp \left\{ \frac{i}{\hbar} \left[ (\epsilon + \zeta_k) t - \frac{a t^3}{3} \right] \right\}. \]

(F.3)

Similarly to what has been done in Appendix C, we employ the normalization condition (C.8), which implies

\[ P_{\text{eff}} = |C_+(\infty)|^2 = \sum_K |U_+^K(\infty)|^2. \]  

(F.4)

Inserting Eq. (F.3) into Eq. (F.4) and using the integral representation of the Airy function [49], we obtain Eq. (4.2).

We next consider the simple case where \( \zeta_k = \zeta, V_k^{(x)} = V, a_k(-\infty) = a_0, \) and \( b_k(-\infty) = b_0 \) for all \( k \). Equation (4.2) then becomes

\[ P_{\text{eff}} = \pi(\delta)^2 \sum_{n=0}^{N} W_n \]

\[ \times \left| Ai[-\tilde{\epsilon}] - \left( \frac{V}{A} \right) \left( n \left( \frac{b_0}{a_0} \right) Ai[-\tilde{\epsilon} + \zeta] + (N-n) \left( \frac{a_0}{b_0} \right) Ai[-\tilde{\epsilon} - \zeta] \right) \right|^2, \]

(F.5)

where \( W_n \equiv |a_0|^{2n}(1 - |a_0|^2)^{N-n} C^N_n \). We next substitute in Eq. (F.5) the following moments of the binomial distribution \( W_n \):

\[ \sum_{n=0}^{N} n W_n = N |a_0|^2, \quad \sum_{n=0}^{N} n^2 W_n = N |a_0|^2 + N(N-1) |a_0|^4, \]

\[ \sum_{n=0}^{N} (N-n)^2 W_n = N^2(1 - |a_0|^2)^2 + N |a_0|^2(1 - |a_0|^2)^2 \]

and

\[ \sum_{n=0}^{N} 2n(N-n) W_n = 2N(N-1) |a_0|^2(1 - |a_0|^2). \]

After some algebra, we finally obtain the expression for \( P_{\text{eff}} \) in Eq. (4.3).
APPENDIX G: EQUATION OF MOTION FOR THE 2 × 2 DENSITY MATRIX

In Section 4(b) we have introduced a coupling of the toy model \((H_0(t), \text{cf. Eq. (2.1)})\) to a general thermal bath, described by the Hamiltonian \((4.4)\). Below we derive an effective equation of motion for the occupation probability of the upper diabatic level, \(P(t)\) (Eq. (4.7)). This derivation generalizes the analysis of Ref. [42]. We obtain explicit expressions for \(B(t)\) and \(C(t)\) in Eq. (4.7), for various types of baths [52]. In particular, we demonstrate the dependence of \(B(t)\) and \(C(t)\) on the spectrum of the bath. In addition, we derive equations of motion similar to Eq. (4.7a) for the off-diagonal elements of the density matrix.

We consider the Heisenberg equations of motion for the operators \(\hat{P}(t), \sigma_\pm(t)\):

\[
\hat{P}(t) = -\frac{i}{\hbar} [H(t), P(t)], \quad \tag{G.1a}
\]

\[
\dot{\sigma}_\pm(t) = -\frac{i}{\hbar} [H(t), \sigma_\pm(t)], \quad \tag{G.1b}
\]

where \(H(t)\) is given by Eq. (4.4). We integrate Eq. (G.1a) and substitute the resulting expression \(\hat{P}(t)\) into its right-hand side. We then trace over the degrees of freedom of the bath and use the thermal averages

\[
\langle F^t(t) \rangle_T = \langle F(t) \rangle_T = 0. \quad \tag{G.2}
\]

For the initial conditions \(P(t_0) = \langle \sigma_\pm(t_0) \rangle_T = 0\) we obtain (to second order in \(A\) and \(V_k\), cf. Eq. (4.4))

\[
\hat{P}(t) = -\left(\frac{A}{2\hbar}\right)^2 \int_{t_0}^{t'} d\tau \langle [\sigma_-(t) + \sigma_+(t), [\sigma_-(\tau) + \sigma_+(\tau), \hat{P}(\tau)]] \rangle_T

-\frac{1}{\hbar^2} \int_{t_0}^{t'} d\tau \langle [F^t(\tau) \sigma_-(t) + F(t) \sigma_+(t), [F^t(\tau) \sigma_-(\tau) + F(\tau) \sigma_+(\tau), \hat{P}(\tau)]] \rangle_T. \quad \tag{G.3}
\]

To second order in \(A\) and \(V_k\), we can use the zeroth order expressions for the operators on the right-hand side of Eq. (G.3). The zeroth order solutions of Eqs. (G.1a) and (G.1b) are, respectively,

\[
\hat{P}(t) \approx \hat{P}(t_0), \quad \tag{G.4a}
\]

and

\[
\sigma_\pm(t) \approx \exp \left\{ \int_{t_0}^{t'} d\tau (\alpha \tau^2 - \varepsilon) / \hbar \right\} \sigma_\pm(t_0); \quad \tag{G.4b}
\]
inserting Eq. (G.4) into Eq. (G.3), we obtain Eq. (4.5). We next employ in Eq. (G.3) the commutation relations
\[ [\sigma_+(t_0), \hat{P}(t_0)] = -\sigma_+(t_0), \quad [\sigma_-(t_0), \hat{P}(t_0)] = \sigma_-(t_0) \] (G.5)
and obtain
\begin{align*}
\hat{P}(t) &\approx -\left(\frac{\Delta}{2\hbar}\right)^2 \int_{t_0}^{t'} dt \langle [e^{-i\phi(t')}\sigma_-(t_0) \\
+ e^{i\phi(t')}\sigma_+(t_0), e^{-i\phi(t)}\sigma_-(t_0) - e^{i\phi(t)}\sigma_+(t_0)] \rangle_T \\
&\quad \times \frac{-1}{\hbar^2} \int_{t_0}^{t'} dt \langle [F^+(t) e^{-i\phi(t)}\sigma_-(t_0) \\
+ F(t) e^{i\phi(t)}\sigma_+(t_0), F^+(\tau) e^{-i\phi(\tau)}\sigma_-(t_0) - F(\tau) e^{i\phi(\tau)}\sigma_+(t_0)] \rangle_T
\end{align*}
where
\[ \phi(t) = \frac{1}{\hbar} \left( \frac{2t^3}{3} - \epsilon t \right) \] (G.6)
To calculate the commutator in Eq. (G.6), we use the operator identity
as well as the commutation relation
\[ [\sigma_+(t_0), \sigma_-(t_0)] = 2\hat{P}(t_0) - 1, \] (G.8)
the relation
\[ \sigma_+(t_0) \sigma_-(t_0) = \hat{P}(t_0), \] (G.9)
and the thermal averages
\[ \langle F^+(t) F^+(\tau) \rangle_T = \langle F(t) F(\tau) \rangle_T = 0, \quad \langle F^+(\tau) F(t) \rangle_T = \langle F^+(\tau - t) F(0) \rangle_T. \] (G.10)
After some algebra, we obtain
\[ \hat{P}(t) \approx -B(t) P(t) + C(t), \] (G.11)
where
\[ B(t) = 2C_0(t) + \frac{1}{\hbar^2} \int_{t_0}^{t'} dt \langle e^{i(\phi(t') - \phi(t))} \{ F^+(t - \tau), F(0) \} \rangle_T + \text{C.C.} \]
and

\[ C(t) = C_0(t) + \frac{1}{\hbar^2} \int_{t_0}^{t} dt' (e^{i\phi(t') - \phi(t)}) \langle F^\dagger (t - \tau) F(0) \rangle \tau + \text{C.C.}; \]

\( C_0(t) \) is defined in Eq. (4.6), the curly brackets denote anti-commutator, and C.C. denotes complex conjugation. We substitute the definitions of \( F^\dagger(t) \) and \( F(t) \) in Eq. (G.11) (cf. Eq. (4.4)), and note that to zeroth order in \( V_k \),

\[ a_k^\dagger(t) = e^{ik(t - t_0)/\hbar} a_k^\dagger(t_0), \quad a_k(t) = e^{-ik(t - t_0)/\hbar} a_k(t_0). \tag{G.12} \]

Replacing the summation over \( k \) by an integration over the spectrum \( \{\zeta_k\} \), Eqs. (G.11) and (G.12) yield Eq. (4.7).

We next define the correlation functions \( K_1(t) \) and \( K_2(t) \),

\[ K_1(t) = \int_{-\infty}^{\infty} d\zeta \exp \left( \frac{i\zeta t}{\hbar} \right) G(\zeta) \langle \{ a^\dagger(\zeta), a(\zeta) \} \rangle, \tag{G.13a} \]

\[ K_2(t) = \int_{-\infty}^{\infty} d\zeta \exp \left( \frac{i\zeta t}{\hbar} \right) G(\zeta) \langle a^\dagger(\zeta) a(\zeta) \rangle. \tag{G.13b} \]

For any physical bath, \( K_1(t) \) and \( K_2(t) \) are Fourier transforms of real functions; hence they satisfy \( K_{1(2)}^*(t) = K_{1(2)}(-t) \). Note that for the particular case where \( a^\dagger(\zeta) \) and \( a(\zeta) \) represent Fermionic operators, the anti-commutator in Eq. (G.13a) is unity. Hence, \( K_1(t) = K(t) \), where \( K(t) \) is the Fourier transform of the power spectrum \( \Gamma(\zeta) \). In the general case, \( K_1(t) \) and \( K_2(t) \) depend on temperature. Below we assume that these correlation functions are vanishingly small, apart from a short time-interval \( \tau_{\text{cor}} \) in the vicinity of \( t = 0 \). We rewrite Eq. (4.7) using the definitions in Eq. (G.13). Defining \( u = t - t_0 \), we obtain

\[ B(t) = 2C_0(t) + \frac{1}{\hbar^2} \left\{ \int_{t_0 - t}^{0} du \exp \left[ \frac{iu}{\hbar} \left( \frac{u^2}{3} \right) \right] K_1(u) + \text{C.C.} \right\}, \tag{G.14a} \]

\[ C(t) = C_0(t) + \frac{1}{\hbar^2} \left\{ \int_{t_0 - t}^{0} du \exp \left[ \frac{iu}{\hbar} \left( \frac{u^2}{3} \right) \right] K_2(u) + \text{C.C.} \right\}. \tag{G.14b} \]

We then impose the condition (4.10), and integrate Eq. (4.7) (with \( B(t) \) and \( C(t) \) given by Eq. (G.14)) over a time interval \( \tau \) which satisfies

\[ \tau_{\text{cor}} \ll \tau \ll \left( \frac{\hbar}{\Delta} \right)^{1/3}. \tag{G.15} \]

Over this time interval, we assume that the change in \( P(t) \) is small and thus obtain

\[ P(t_0 + \tau) - P(t_0) \approx - \int_{t_0}^{t_0 + \tau} dt \, B(t) \, P(t_0) + \int_{t_0}^{t_0 + \tau} dt \, C(t). \tag{G.16} \]
We insert Eq. (G.14) into Eq. (G.16) and change the order of integration (i.e., integrate first over $t$ for a fixed $u$):

$$P(t_0 + \tau) - P(t_0) \approx (1 - 2P(t_0)) \int_{t_0}^{t_0 + \tau} dt \, C_0(t)$$

$$+ \frac{1}{\hbar^2} \left\{ \int_{-\tau}^{\tau} du \, K_2(u) \exp \left\{ \frac{iu}{\hbar} \left[ \epsilon - \frac{zu^2}{12} \right] \right\} \right.$$  

$$\times \int_{t_0 - u}^{t_0 + \tau} dt \exp \left\{ -\frac{izu}{\hbar} \left( t + \frac{u}{2} \right)^2 \right\} + \text{C.C.} \}$$

$$- \frac{1}{\hbar^2} \left\{ \int_{-\tau}^{\tau} du \, K_1(u) \exp \left\{ \frac{iu}{\hbar} \left[ \epsilon - \frac{zu^2}{12} \right] \right\} \right.$$  

$$\times \int_{t_0 - u}^{t_0 + \tau} dt \exp \left\{ -\frac{izu}{\hbar} \left( t + \frac{u}{2} \right)^2 \right\} + \text{C.C.} \} P(t_0). \tag{G.17}$$

The integration over $t$ in the last two terms yields

$$\int_{t_0 - u}^{t_0 + \tau} dt \exp \left\{ -\frac{izu}{\hbar} \left( t + \frac{u}{2} \right)^2 \right\}$$

$$= \frac{1}{2} \sqrt{\frac{\pi \hbar}{-izu}} \left\{ \text{erf} \left[ \sqrt{-\frac{izu}{\hbar}} \left( t_0 + \tau + \frac{u}{2} \right) \right] - \text{erf} \left[ \sqrt{-\frac{izu}{\hbar}} \left( t_0 - \frac{u}{2} \right) \right] \right\}. \tag{G.18}$$

Employing Eq. (G.15), we expand the right-hand side of Eq. (G.18) near $t_0$ and obtain

$$P(t_0 + \tau) - P(t_0)$$

$$\approx (1 - 2P(t_0)) \int_{t_0}^{t_0 + \tau} dt \, C_0(t)$$

$$+ \frac{1}{\hbar^2} \left\{ \int_{-\tau}^{\tau} du \, K_2(u) \exp \left\{ \frac{iu}{\hbar} \left[ \epsilon - zt_0^2 - \frac{zu^2}{12} \right] \right\} \right.$$  

$$(\tau + u) + \text{C.C.} \}$$

$$- \frac{1}{\hbar^2} \left\{ \int_{-\tau}^{\tau} du \, K_1(u) \exp \left\{ \frac{iu}{\hbar} \left[ \epsilon - zt_0^2 - \frac{zu^2}{12} \right] \right\} \right.$$  

$$(\tau + u) + \text{C.C.} \} P(t_0). \tag{G.19}$$

Equation (G.15) implies that one can assume $u \ll \tau$ in the above integrals, since $K_1(u)$ and $K_2(u)$ are vanishingly small for $|u| > \tau_{\text{cor}}$. This property of the correlation functions also implies that the lower limit in the integrals, $(-\tau)$, may be safely replaced by $-\infty$. In addition, we neglect the slowly oscillating term $(zu^3/12\hbar)$ in the exponents (employing Eq. (G.15)). We thus verify that the last two terms in
Eq. (G.19) are proportional to the inverse Fourier transforms of $K_1(t)$ and $K_2(t)$, with respect to $(\alpha t_0^2 - \varepsilon)/\hbar$. Dividing Eq. (G.19) by $\tau$ and using

$$
\frac{P(t_0 + \tau) - P(t_0)}{\tau} \approx \hat{P}(t_0), \quad \text{(G.20)}
$$

we obtain an equation which recovers Eq. (4.7a), with $B(t)$ and $C(t)$ given by

$$
B(t) \approx 2C_0(t) + \frac{1}{\hbar^2} G(\alpha t^2 - \varepsilon) \langle \{a^\dagger(\alpha t^2 - \varepsilon), a(\alpha t^2 - \varepsilon)\} \rangle_T, \quad \text{(G.21a)}
$$

$$
C(t) \approx C_0(t) + \frac{1}{\hbar^2} G(\alpha t^2 - \varepsilon) \langle a^\dagger(\alpha t^2 - \varepsilon) a(\alpha t^2 - \varepsilon) \rangle_T. \quad \text{(G.21b)}
$$

We now employ in Eq. (G.21a) the canonical anti-commutation (commutation) relations for a bath of fermions (bosons). In addition, we use the thermal averages

$$
\langle a^\dagger(E) a(E) \rangle_T = \frac{1}{e^{\beta E} + 1} \quad \text{(fermions)}, \quad \text{(G.22a)}
$$

$$
\langle a^\dagger(E) a(E) \rangle_T = \frac{1}{e^{\beta E} - 1} \quad \text{(bosons)}. \quad \text{(G.22b)}
$$

Inserting into Eq. (G.21), we obtain Eqs. (4.11) and (4.12). Finally, we consider the case where the bath consists of classical degrees of freedom. In this case

$$
\langle \{a^\dagger(E), a(E)\} \rangle_T = 2 \langle a^\dagger(E) a(E) \rangle_T, \quad \text{(G.23)}
$$

which imply Eq. (4.13).

To complete the discussion of dissipation in the toy model, we derive below an effective equation of motion (similar to Eq. (4.7)) for the off-diagonal elements of the density matrix. These correspond to the expectation values $\langle \sigma_\pm(t)/2 \rangle_T$. The Heisenberg equations of motion (G.1b), to first order in $\Delta$ and the coupling strength, are

$$
\dot{\sigma}_\pm(t) = \pm \frac{i(\alpha t^2 - \varepsilon)}{\hbar} \sigma_\pm(t) \pm \frac{i\Delta}{2\hbar} \left(1 - 2\hat{P}(t)\right) - \frac{i}{\hbar} \left[V(t), \sigma_\pm(t)\right], \quad \text{(G.24)}
$$

where $V(t)$ is defined in Eq. (4.5). We introduce a gauge transformation

$$
\tilde{\sigma}_\pm(t) \equiv e^{i\phi(t)} \sigma_\pm(t), \quad \text{(G.25)}
$$

where $\phi(t)$ is defined in Eq. (G.6). Similarly to the procedure used in the derivation of Eq. (4.5), we integrate Eq. (G.24) and substitute the resulting expression for $\sigma_\pm(t)$ into the right-hand side of Eq. (G.24). After tracing over the bath degrees of
freedom and imposing the initial conditions \( P(t_0) = \langle \sigma_+ (t_0) \rangle_T = 0 \), we obtain to second order in \( V(t) \) and first order in \( A \),

\[
\langle \dot{\sigma}_+ (t) \rangle_T \approx \frac{iA}{2\hbar} e^{-i\phi(t)} e^{-i\phi(t)} e^{i\phi(t)}
\]

\[
- \frac{1}{\hbar^2} \int_{t_0}^{t'} d\tau \langle [F(t) \sigma_+(t_0) e^{i\phi(t)} + F^\dagger(t) \sigma_-(t_0) \exp^{-i\phi(t)} F^\dagger(\tau) e^{-i\phi(t)} (1 - 2\tilde{P}(t_0))] \rangle_T , \quad \text{(G.26a)}
\]

\[
\langle \dot{\sigma}_- (t) \rangle_T \approx -\frac{iA}{2\hbar} e^{i\phi(t)} e^{i\phi(t)} e^{-i\phi(t)} \exp^{-i\phi(t)}
\]

\[
- \frac{1}{\hbar^2} \int_{t_0}^{t'} d\tau \langle [F(t) \sigma_+(t_0) e^{i\phi(t)} + F^\dagger(t) \sigma_-(t_0) \exp^{-i\phi(t)} F(\tau) e^{i\phi(t)} (2\tilde{P}(t_0) - 1)] \rangle_T ; \quad \text{(G.26b)}
\]

here we have used the explicit form of \( V(t) \) (cf. Eq. (4.5)) and Eqs. (G.2), (G.8). Employing Eqs. (G.5), (G.7), (G.9), (G.10), and

\[
\sigma_+^2 (t_0) = \sigma_-^2 (t_0) = 0, \quad \text{(G.27)}
\]

we obtain

\[
\langle \dot{\sigma}_\pm (t) \rangle_T \approx \pm \frac{iA}{2\hbar} e^{\pm i\phi(t)}
\]

\[
- \frac{1}{\hbar^2} \int_{t_0}^{t'} d\tau e^{\pm i(\phi(t) - \phi(\tau))} \langle \{ F^\dagger (\pm (t - \tau)), F(0) \} \rangle_T \langle \dot{\sigma}_\pm (t) \rangle. \quad \text{(G.28)}
\]

We then insert into Eq. (G.28) the explicit forms for \( F^\dagger (t) \) and \( F(t) \); similarly to what has been done for the diagonal element \( P(t) \), we change \( \sum_\xi \) into an integral over the spectrum. We thus obtain equations of motion which are similar to Eq. (4.7):

\[
\langle \dot{\sigma}_\pm (t) \rangle_T = -B_\pm (t) \langle \dot{\sigma}_\pm (t) \rangle_T + C_\pm (t), \quad \text{(G.29)}
\]

where

\[
C_\pm (t) = \pm \frac{iA}{2\hbar} e^{\mp i\phi(t)},
\]

\[
B_\pm (t) \approx \frac{1}{\hbar^2} \int_{t_0}^{t'} d\tau e^{\pm i(\phi(t) - \phi(\tau))} K_1 (t - \tau);
\]
\( K_1(t) \) is defined in Eq. (G.13a). Following the same procedure employed above to derive Eq. (G.21), we obtain

\[
B_+(t) \approx \int_{-\infty}^{0} du \exp \left\{-\frac{i}{\hbar} (\alpha t^2 - \epsilon)u \right\} K_1(u),
\]

\[
B_-(t) \approx \int_{0}^{\infty} du \exp \left\{-\frac{i}{\hbar} (\alpha t^2 - \epsilon)u \right\} K_1(u).
\]

Equation (G.30) implies that in the particular case where \( K_1(u) \) is a symmetric function \( (K_1(u) = K_1(-u)) \), \( B_+(t) = B_-(t) = B(t)/2 \).

**APPENDIX H: DIFFUSION EQUATION FOR THE EVOLUTION IN THE ENERGY SPACE**

We consider the dynamics of the quantum mechanical system, described by Eq. (5.2), in the presence of external degrees of freedom which provide both dephasing and dissipation. Under certain assumptions, discussed in Section 5, the time evolution of the system is determined by a master equation for the density matrix. Within each half a period of the adiabatic spectrum, the dynamics of the density matrix is given by Eqs. (5.3a) and (5.8). Below we consider Eq. (5.8) in the limit \( \tau_\phi \ll \tau_0 \) and derive a diffusion equation with a relaxation term for the diagonal density matrix elements (Eq. (5.14)). We then derive the steady state solution of this equation.

In the limit \( \tau_\phi \ll \tau_0 \), the off-diagonal elements of the density matrix can be neglected. We hence focus on the dynamics of the diagonal matrix elements \( \rho_n(t) \), during each period of the spectrum—e.g., the interval \( t^- \leq t < t^+ \) in Fig. 4. Equation (5.3a) implies

\[
\rho_n(t^+) = T\rho_{n-1}(t^-) + R\rho_n(t^-);
\]

here \( n \) represents an even level (relative to the Fermi level) and \( t = m\tau_0/2 \) with \( m \) an even integer as well; \( t^\pm = t \pm \tau_2 \), \( T = |\bar{t}|^2 \), and \( R = |\bar{r}|^2 \), where \( \bar{t} \) and \( \bar{r} \) are the transmission and reflection amplitudes, respectively, associated with each narrow gap in the spectrum. Inserting Eq. (H.1) into Eq. (5.8) we obtain

\[
\rho_n(t^-) = \rho_n^{eq} + \left\{ T\rho_{n-1}(t^-) + R\rho_n(t^-) - \rho_n^{eq}\right\} e^{-\tau_0/2\tau_{eq}}.
\]

Similarly, for \( \rho_{n+1}(t) \) at \( t = t^- \) we obtain

\[
\rho_{n+1}(t^-) = \rho_{n+1}^{eq} + \left\{ T\rho_{n+2}(t^-) + R\rho_{n+1}(t^-) - \rho_{n+1}^{eq}\right\} e^{-\tau_0/2\tau_{eq}}.
\]

At \( t_m \) (see Fig. 4) another scattering event takes place, whence

\[
\rho_n(t^+) = T\rho_{n+1}(t_m^-) + R\rho_n(t_m^-).
\]
Inserting Eqs. (H.2) and (H.3) into Eq. (H.4) and employing Eq. (5.8) once more, we obtain

\[
\rho_n(t_f^-) = \rho_n^{\text{eq}} + \left\{ (T \rho_{n+1}^{\text{eq}} + R \rho_n^{\text{eq}})(1 - e^{-\tau_0/2\tau_{eq}}) - \rho_n^{\text{eq}} \right\} e^{-\tau_0/2\tau_{eq}} \\
+ \left\{ T^2 \rho_{n+2}(t_i^-) + R^2 \rho_n(t_i^-) + RT(\rho_{n+1}(t_i^-) + \rho_{n-1}(t_i^-)) \right\} e^{-\tau_0/\tau_{eq}}. \quad (H.5)
\]

We now assume \( T \ll 1 \) (small Zener probability), and \( \tau_{eq} \gg \tau_0 \). In this limits one can expand the exponents in Eq. (H.5) to linear order in \( \tau_0/\tau_{eq} \) and neglect terms of orders \( T^2, T\tau_0/\tau_{eq} \). We thus obtain

\[
\rho_n(t_f^-) - \rho_n(t_i^-) = \rho_n(t_i^- + \tau_0) - \rho_n(t_i^-) \\
\approx \left\{ (R^2 - 1) \rho_n(t_i^-) + RT(\rho_{n+1}(t_i^-) + \rho_{n-1}(t_i^-)) \right\} - \rho_n^{\text{eq}} \frac{\tau_0}{\tau_{eq}} \\
\approx T\rho_{n+1}(t_i^-) - 2\rho_n(t_i^-) + \rho_{n-1}(t_i^-) \frac{\rho_n^{\text{eq}}}{\tau_{eq}}; \quad (H.6)
\]

to obtain the last equality we use the unitarity of the scattering matrix \((R + T = 1)\). We then divide Eq. (H.6) by the period \( \tau_0 \) and note that, for small \( \tau_0 \), \((\rho_n(t + \tau_0) - \rho_n(t))/\tau_0 \approx \dot{\rho}_n(t)\). As a result we obtain Eq. (5.14); the expression for the diffusion coefficient \( D(V) \) is obtained using Eq. (2.8) for the transition amplitudes, where \( P_{LZ} \) is given by Eq. (2.6).

We next derive the steady state solution of Eq. (5.14), \( \rho_n^{ss} \), which satisfies \( \dot{\rho}_n(t) = 0 \):

\[
D(V)(\rho_n^{ss}_{n+1} - 2\rho_n^{ss} + \rho_n^{ss}_{n-1}) = \frac{\rho_n^{ss} - \rho_n^{eq}}{\tau_{eq}}. \quad (H.7)
\]

We first note that the solution for the homogeneous equation, defined as the equation obtained from (H.7) replacing \( \rho_n^{eq} \) by zero, is of the form

\[
\rho_n^H = \rho_0 e^{-\eta n} \quad (H.8)
\]

(which satisfies the boundary conditions at \( n \to \infty \)). A direct substitution of Eq. (H.8) into the homogeneous equation results in

\[
\eta = 2 \sinh^{-1}(1/2 \sqrt{D(V)\tau_{eq}}). \quad (H.9)
\]

According to our underlying assumptions, \( \rho_n^{eq} \) is also a simple exponential function of \( n \) (cf. Eq. (5.7)). Hence, we try a general solution for (H.7) of the form

\[
\rho_n^{ss} = \rho_n^H + C \rho_n^{eq}. \quad (H.10)
\]
Here $C$ is a constant parameter, to be determined by insertion of Eq. (H.10) and (5.7) into Eq. (H.7); we find

$$C = \frac{1}{1 - 4D(V) \tau_{eq} \sinh^2(\zeta/2)}.$$  \hspace{1cm} (H.11)

We next require the normalization

$$\sum_{n=1}^{\infty} \rho_n^{ss} = 1,$$  \hspace{1cm} (H.12)

and thus determine $\rho_0$ (cf. Eq. (H.8)). Inserting Eqs. (H.8), (H.9), and (H.11) into Eq. (H.10), we obtain the explicit expression for the steady state solution in Eq. (5.15).

ACKNOWLEDGMENTS

We gratefully acknowledge discussions with Professors Zeev Schuss, Michael Berry, Ady Stern, Eshel Ben-Jacob, Doron Cohen, and Shmuel Fishman. This research was supported by the U.S.-Israel Binational Science Foundation, the Minerva Foundation, Munich, FRG, and the German–Israeli Foundation for Scientific Research and Development.

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21. Such a dissipative relaxation, in addition to Zener transitions, was included in the description of a driven Josephson junction by A. D. Zaikin and I. N. Kosarev, Phys. Lett. A 131 (1988), 125.
34. We make here an assumption that this asymptotic probability approximates the actual transition probability at each narrow gap.
36. Note that Eq. (2) does not hold for \( \theta \neq 0 \).
37. We thank Dr. Doron Cohen for drawing our attention to this analogy.
38. The situation is different in the Landau–Zener case, where the coefficient of \( \sigma_z \) in the Hamiltonian is linear in time. In that case, we have a distribution of shifts of the avoided crossing point. Since the Landau–Zener probability \( P_{LZ} \) is independent of the crossing point, the effective transition probability (obtained by replacing \( P \) in Eq. (3.6) by \( P_{LZ} \)), is not affected by the environment.
39. In order to give rise to transitions among adiabatic states of the system (and therefore to dissipation), one should couple the system to the bath through the operators \( \sigma_x, \sigma_y \) (see Section 4).
41. E. Shimshoni and A. Stern, unpublished.
43. A continuous time dependence, when \( \rho^{(n)} \) is taken to be time dependent (cf. Eq. (5.6)), will modify this latter feature.
44. E. Shimshoni and Y. Gefen, unpublished.
45. We note that our model Hamiltonian (Eq. (5.2)) describes, in principle, also a small Josephson junction driven by a dc current source. However, in the latter example the narrow gaps between consecutive levels in the spectrum decrease exponentially, hence localization does not occur even in the purely quantum coherent case. Localization effects are an important ingredient in the rich and complex behavior described in the present work.


51. Another generalization of Ref. [22], for the case where the bias is linear in time (ε(t) = at), was considered by Dorsey in Ref. [40].

52. We note that our derivation follows closely the derivation of the classical Langevin equation; see, e.g., R. Reif, "Fundamentals of Statistical and Thermal Physics," Chap. 15, McGraw–Hill, New York, 1965.