Condensation in temporally correlated zero-range dynamics

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• Classical condensation phenomena
• Condensation transition in the zero-range process
• Non-Markovian ZRP dynamics
• Conclusions
1. Some classical condensation phenomena

**Granular shaking:**

N=100 plastic particles in box with two compartments separated by wall with slit

[Schlichting and Nordmeier ‘96, Eggers ‘99, Lohse ‘02]

\[ T > T_c \] \hspace{1cm} \text{Gaseous state} \hspace{1cm} \text{T > T}_c

\[ T < T_c \] \hspace{1cm} \text{Condensed state} \hspace{1cm} \text{T < T}_c

i) Strong shaking (fixed amplitude, 50 Hz frequency): \( \rightarrow \) Equal gaseous distribution

ii) Moderate shaking (same amplitude, 30 Hz): \( \rightarrow \) Condensation (with SSB)

Effective, frequency-dependent temperature leads to phase transition
Granular Clustering: $L=5$

http://stilton.tnw.utwente.nl/people/rene/clustering.html

Detlef Lohse, Devaraj van der Meer, Michel Versluis,
Ko van der Weele, René Mikkelsen

Time $t = 0 \ldots 12$ sec

$t$ approx. 1 min
**Single File Diffusion:**

SFD: Quasi one-dimensional diffusion without passing

- diffusion in zeolites
- colloidal particles in narrow channels
- ion channels
- molecular motors and ribosomes
- gel electrophoresis
- one-dimensional interface growth
- automobile traffic flow
- ...

**Condensation = traffic jam = phase separation**

[Three phases of kinesin transport (Chodhury et al.)](http://omega.dawsoncollege.qc.ca/ray/protein/protein.htm)
2. Condensation transition in the zero-range process

Zero-range process (ZRP) with symmetric nearest-neighbour hopping [Spitzer (1970)]

- Stochastic microscopic particle hopping model for large scale hydrodynamic behaviour
- Cluster of size \(n\) \(\Leftrightarrow\) occupation number in ZRP
- Particle flux \(J(n_k)\) between compartments \(\Leftrightarrow\) hopping rate in ZRP
Mapping of single-file diffusion to zero range process:

- Label particles consecutively

- Map particle label to lattice site

- Map discretized interparticle distance to particle number
Condensation transition ➔ Proposed to explain condensation phenomena

• Granular shaking
• Network rewiring
• Accumulation of wealth

Mapping to single-file dynamics (one-dimensional):
• Phase separation in multi-component systems
• Traffic flow

Generic model for condensation in complex systems
Exact grand canonical stationary distribution of zero-range process [Spitzer, (1970)]

Product measure with marginals $P(n)$ and local partition function $Z$

$$P(\mathbf{n}) = \prod_{i \in \Lambda} P(n_i)$$

$$P(n) = \frac{1}{Z} z^n \prod_{k=1}^{n} J^{-1}(k), \quad Z = \sum_{n=0}^{\infty} \tilde{P}(n)$$

- Fugacity $z$ determines (fluctuating) density
- Well-defined for fugacities within radius of convergence $z^*$ (that depends on flux function)
- Canonical ensembles for any $N$ by projection on fixed $N$
Spatially homogeneous systems

1) Asymptotically vanishing flux $J(n) \to 0$: $\Rightarrow z^* = 0$ and hence $\rho_c = 0$ (strong condensation)

2) Consider generic case where for large $n$

$$J(n) = A \left(1 + \frac{b}{n^\sigma}\right)$$

$\Rightarrow$ radius of convergence of partition function: $z < z^* = A$

$\Rightarrow$ at $z^*$ one has finite density $\rho_c$ for $\sigma < 1$

$\Rightarrow$ For $\sigma = 1$: $\Rightarrow P(n) \sim 1/n^b$

$$\rho(z^*) = \begin{cases} \infty & \text{for } b \leq 2 \\ \rho_c = \frac{1}{(b - 2)} & \text{for } b > 2 \end{cases}$$
Interpretation of critical density for $b>2$ or $\sigma < 1$ for canonical ensemble:

- Above critical density all sites except one (background) are at critical density
- One randomly selected site carries remaining $O(L)$ particles

- **Classical analogue of Bose-Einstein condensation**
  [Evans '96, Ferrari, Krug '96, O'Loan, Evans, Cates, '98, Jeon, March '00]

- **Single random condensation site**
  [Grosskinsky, GMS, Spohn, '05, Ferrari, Landim, Sisko '07, Loulakis, Armendariz '08, Evans, Majumdar '08]

- **Continuous condensation transition** ($\rho_{bg} = \rho_c$)

- **Coarsening as precursor of condensation**
  [Grosskinsky, GMS, Spohn, '05; Godreche '05]
Remarks:

- Product measure stationary for ZRP on arbitrary graph

- Single-file dynamics ($n = \text{interparticle distance}$) ⇒ 1d phase transition?
  
  Thermally activated jumps: $J(n) \sim \exp(-\beta E(n)) \sim \exp\left(b/n^\sigma\right)$

  $E(n) = a + b/n^\sigma + ...$ ⇒ Long range interaction in 1d!

- Basic mechanism of condensation:

  Growth of large domains on the expense of small domains

  ⇒ Asymptotically decaying $J(n)$ with critical decay exponent $\sigma = 1$

  ⇒ In this case, condensation depends on interaction strength $b$
3. Non-Markovian ZRP dynamics

Complex systems: Markovian property (lack of memory) may be unjustified
(e.g. colloidal particles in a fluid: power law tail in velocity autocorrelation)

- Introduce memory term (on microscopic level)

- Is condensation stable w.r.t. memory?
- Can memory induce condensation?

Example: AHR model for probe particle in a driven fluid:
- strongly correlated non-Markovian jumps with effective jump rate
- domain size distribution (distance between probes) identical to ZRP
- no condensation, but “almost” (huge mean domain size)
Our approach to model non-Markovian dynamics:

- make jump rates dependent on “age” of site i (integer clock $\tau_i$) $\Rightarrow u(n,\tau)$
- age measured since last arrival (reset $\tau(k) = 0$ at arrival of particle)
- discrete increments $\tau_i \rightarrow \tau_i + 1$ at exponential random times
- clock increment independent of $n_i$, but in general depending on other clocks

$\Rightarrow$ Joint dynamics $(n(k),\tau(k))$ is Markovian

$\Rightarrow$ Particle hopping $n(k)$ by itself is non-Markovian and zero range
1) Special case: On-off model with interaction of clocks

- Consider on-off case $\tau = 0,1$
- Asymmetric nearest neighbour jumps

$$u(n, \tau) = \begin{cases} 
0 & \tau = 0 \quad \text{("off" state)} \\
u(n) & \tau \geq 1 \quad \text{("on" state)}
\end{cases}$$

- Clock increment depending on target site

**Exact results:**
- Stationary distribution factorizes into-site marginals $P(n) = P_0(n) + P_1(n)$
- $P(n)$ same form as Markovian ZRP with effective hopping rate

$$u_{\text{eff}}(n) = c \frac{u(n)}{c + u(n)}$$

$\Rightarrow$ Shift in critical $b$ for condensation
2) Generic model without clock interaction

- make jump rates dependent on “age” of site i (integer clock $\tau_i$) 
  $\Rightarrow u(n, \tau)$

- age measured since last arrival (reset $\tau(k) = 0$ at arrival of particle)

- discrete increments $\tau_i \rightarrow \tau_i + 1$ at exponential random times (independent of $n_i$ and other clocks)

\[
(n_i, \tau_i) \xrightarrow{u(n_i, \tau_i)} (n_i - 1, \tau_i), (n_i + 1, \tau_j = 0)
\]

\[
(n_i, \tau_i) \xrightarrow{c} (n_i, \tau_i + 1),
\]

Consider two cases:

A) Mean field dynamics: Uniform random target site j (fully connected graph)

B) Totally asymmetric nearest neighbour dynamics (1-d periodic lattice)
A) Mean field dynamics:

- Uniform random target site $j$: Mean Field (MF) dynamics
- approximate factorization for large $L$
- focus on single site with incoming “mean-field” current $J$

\[
\frac{dP(n, \tau)}{dt} = -P(n, \tau)\left[J + c + u(n, \tau)\right] + J\delta_{\tau,0}P(n-1) + cP(n, \tau-1) + u(n+1, \tau)P(n+1, \tau)
\]

with average occupation number $P(n) \equiv \sum_\tau P(n, \tau)$

and current $\bar{J} = \sum_{n, \tau} u(n, \tau)P(n, \tau)$
Stationary distribution:

- set time-derivative to zero
- define mean hopping rate

\[ \bar{u}(n) \equiv \frac{\sum_\tau P(n, \tau) u(n, \tau)}{\sum_\tau P(n, \tau)} \]

\[ P(n) = P(0) J^n \bar{f}(n) \quad \text{with} \quad \bar{f}(n) = \prod_{i=1}^{n} \bar{u}(i)^{-1} \]

\[ \text{Same form as usual Markovian ZRP with hopping rate } \bar{u}(n) \]

and current \( J=z \)
Shift of condensation transition:

- Critical current

\[ J_c = \frac{c}{2} \left( \sqrt{1 + \frac{4}{c}} - 1 \right) \]

===> \( b \) has to be larger than 2, condensation transition for

\[ b > \frac{4}{c} \left( \sqrt{1 + \frac{4}{c}} - 1 \right)^{-1} \]

Memory destroys condensation for \( b \) close to 2
On-off model:

- Consider on-off case $\tau = 0,1$

$$u(n, \tau) = \begin{cases} 
0 & \tau = 0 \text{ ("off" state)} \\
u(n) & \tau \geq 1 \text{ ("on" state)}
\end{cases}$$

- Mean hopping rate

$$\frac{1}{\bar{u}(n)} = \frac{P_{\text{off}}}{J} + \frac{1}{u(n)}$$

$$\bar{u}(n) \sim \frac{c + J}{c + J + 1} \left(1 + \frac{b_{\text{eff}}}{n}\right)$$

Same form as usual Markovian ZRP with interaction parameter $b_{\text{eff}}$
B) Totally asymmetric on-off model with periodic boundary conditions:

- mean field approximation not good
- condensate typically occupies two sites
- condensate moves
Motion of condensate:

- position of most occupied site
- occupation of most occupied site

\[ \text{speed } v = \frac{1}{N-N_c} \sim \frac{1}{L} \]
Conclusions

1. Construction of family ZRP with memory

2. Exactly solvable case with coupled clocks: product measure, modified ZRP hopping rates that affect condensation

3. General mean field dynamics with uncoupled clocks: modified ZRP hopping rates that affect condensation (large L)

4. Totally asymmetric On-off model with nearest neighbour hopping in one dimension:
   - condensate occupies two sites
   - slinky motion with finite velocity $\sim 1/L$

$\Rightarrow$ Similar conclusions for heterogeneous Single-File Diffusion with long range interaction
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